An Interior Point Algorithm for Nonlinear Programming

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Abstract

This paper describes an interior point algorithm for solving nonlinear programming problems. The approach we consider here is a classical one consisting of transforming the original problem in one with only equality constraints, the inequalities being placed into the objective through a logarithmic barrier function. The KKT optimality conditions are solved by the Newton method. The direction of moving is given by a linear algebraic system, which must be solved at each iteration. The step length computation, which is the critical point of the algorithm, is based on a merit function which determine the values of the barrier parameter which ensure the descendence character of the search direction. The conditions for step length computation include: the boundary of variables, the positivity of the slack and dual variables, the centrality of iterations, the corelation between the speed of deacreasing of the pure optimality conditions and the transversality conditions, as well as the Wolfe conditions.
A crude implementation of the algorithm shows the performance of this approach on a number of nonlinear problems from the Hock-Schittkowski set of problems. Comparisons with a modified penalty-barrier algorithm implemnted as SPENBAR package show that the method is efficient on at least some classes of nonlinear models.

1. Introduction.

Interior point methods for nonlinear programming is one of the most active research area in mathematical programming. This is motivated by the impressive high quality theoretical results and computational performance of these methods for linear and quadratic programming, as well as for linear constrained optimization. We can mention here, for example, the papers: [Andersen and Andersen, 1997], [Andrei, 1998b], [Carpenter, Lustig, Mulvey and Shanno, 1993], [Gondzio, 1996], [Lustig, Marsten and Shanno, 1990, 1991, 1992, 1994], [Vanderbei,1990, 1997], [Ye, 1997].

Having in view these remarkable results, it is natural that the researches using interior point ideas have been directed to the nonlinear programming area, [Kortanek, Potra and Ye, 1991], [Goldfarb, Liu and Wang, 1991], [Wright, 1992, 1998].

We are interested in solving the following nonlinear programming problem:

$$\begin{aligned} &min f(x) \\ &subject\ to: \\ &g(x) = 0, \\ &h(x) \geq 0, \\ &l \leq x \leq u, \end{aligned} \tag{1.1}$$

where $f : R^n \to R$, $g : R^{me} \to R$ and $h : R^m \to R$ are supposed to be twice continuous differentiable. The bounds l and u may have any values with $l < u$, in practice these could be $\pm\infty$. For solving this general nonlinear programming problem we have a number of methods and algorithms which could be classified as: Direct Methods, Methods based on KKT Optimality Conditions, Penalty Methods, and Interior Point Methods. The interior point methods have recently been the subject of a number of papers. The formulation and theory of the Newton interior point method for nonlinear programming is given in [El-Bakry, Tapia, Tsuchiya and Zhang, 1996]. Gay, Overton and Wright [1997] consider a primal-dual method for nonconvex programming in which the inequality constraints are introduced into the objective function by means of the classical logarithmic barrier function, and the equality constraints are considered into a sequential quadratic programming mechanism. Vanderbei and Shanno [1997] transform the original inequality problem to a equality one by adding slack to each of the inequality constraints, for which the first order conditions for a minimum are solved by means of a Newton strategy. A special method for choosing the step length along the Newton direction is introduced. This is based on a merit function. An extension of LOQO package to consider the nonlinear function, interfaced with AMPL [Fourer, Gay and Kernighan, 1993] and GAMS [Brooke, Kendrick and Meeraus, 1992], is illustrated on the Hock and Schittkowski suite, Mittelmann's quadratic programming set, as well as on a number of 8 large-scale real-world problems. Some comparisons and performances of LOQO with MINOS [Murtagh and Saunders, 1995] and LANCELOT [Conn, Gould and Toint, 1992] are presented. Byrd, Hribar and Nocedal [1997] incorporates within the interior point method two powerful tools for solving nonlinear problems: the sequential quadratic programming and the trust region techniques. The NITRO package is presented and its performance on some numerical examples is illustrated. Some other primal-dual interior methods for nonconvex nonlinear optimization problems are discussed in [Argaez and Tapia, 1997], [Byrd, Gilbert and Nocedal, 1996], [Conn, Gould and Toint, 1996], [Forsgren and Gill, 1996], etc.

The algorithms are based on the classical approach: a minimizing direction is computed and than a step length on this direction is considered. When the problem is a *convex program* the step length can be determined by standard line search methods, which are direct extensions of interior point methods for linear programming. In this case the step generated by the solution of the primal-dual equations is a descent direction for different merit functions, thus ensuring the convergence of the corresponding algorithms. For *non-convex programming*, the method for computing the step length at each iteration is more complex. In fact, the notably differences between algorithms is the way in which the step length is computed. Thus, for examples, Vanderbei and Shanno [1997] introduce into the merit function a parameter which is updated as soon as the computed direction is not a descent direction. On the other hand, Gay, Overton and Wright [1997] describe a heuristical procedure combined with a simple Armijo-like rule to determine the step length.

In this paper we shall present an interior point predictor corrector algorithm for solving (1.1) which is based on the theoretical developments given in [El-Bakry, Tapia, Tsuchiya and Zhang, 1996]. Basically, for the considered problem we write down the KKT optimality conditions. Using the Newton method we solve these optimality conditions with respect of the moving direction. Considering a suitable merit function the updating conditions for the barrier parameter are established. The mechanism for step length determination is based on the Wolfe conditions combined with the interior point centrality condition and the limitation of the convergence to zero of the transversality conditions against the "pure primal-dual" optimality condition. The paper is organized as follows. Section 2 is devoted to present the KKT optimality conditions. The Newton system and the Newton direction are considered in the sections 3 and 4, respectively. The merit function is introduced in section 5. Section 6 is devoted to compute the step length on the Newton direction. The next sections present the primal-dual algorithm and some numerical examples.

2. KKT Optimality Conditions.

Considering the slack variables $s \in R^m$, associated to the functional inequality constraints and $w, v \in R^n$ associated to the simple bounds of the variable x, the problem (1.1) can be reformulated as:

$$\min f(x)$$

$$
\begin{aligned}
&subject\ to: \\
&g(x) = 0, \\
&h(x) - s = 0, \\
&x - w = l, \\
&x + v = u, \\
&s \geq 0, w \geq 0, v \geq 0.
\end{aligned} \tag{2.1}
$$

Now, we eliminate the inequality constraints in (2.1) by placing them into a barrier term, obtaining the problem:

$$
\begin{aligned}
\min f(x) - \mu \sum_{i=1}^{m} \log s_i - \mu \sum_{j=1}^{n} \log w_j - \mu \sum_{j=1}^{n} \log v_j \\
subject\ to: \\
g(x) = 0, \\
h(x) - s = 0, \\
x - w - l = 0, \\
x + v - u = 0,
\end{aligned} \tag{2.2}
$$

where the objective function in (2.2) is the classical Fiacco-McCormick [1968] logarithmic barrier function.

The Lagrangian for this problem is:

$$
\begin{aligned}
L(x, s, w, v, y, z, p, q, \mu) = \\
f(x) - \mu \sum_{i=1}^{m} \log s_i - \mu \sum_{j=1}^{n} \log w_i - \mu \sum_{j=1}^{n} \log v_i - \\
y^T g(x) - z^T (h(x) - s) - p^T (x - w - l) + q^T (x + v - u),
\end{aligned} \tag{2.3}
$$

where y, z, p and q are the dual variables of the corresponding dimensions. With these, the first order optimality conditions are:

$$
\begin{aligned}
\nabla f(x) - \nabla g(x)^T y - \nabla h(x)^T z - p + q = 0, \\
g(x) = 0, \\
h(x) - s = 0, \\
x - w - l = 0, \\
x + v - u = 0, \\
SZe - \mu e = 0, \\
WPe - \mu e = 0, \\
VQe - \mu e = 0,
\end{aligned} \tag{2.4}
$$

where the matrices S, Z, W, P, V and Q are diagonal with elements s_i, z_i, w_i, p_i, v_i and q_i respectively, e is the vector of all ones. $\nabla f(x)$ is the gradient of the objective function $f(x)$. $\nabla g(x)$ and $\nabla h(x)$ are the Jacobian matrices of the constraints vectors $g(x)$ and $h(x)$ respectively.

Note that the above optimality conditions (known as KKT conditions) contain the constraints of the original problem, an equality reseambling of the dual problem, as well as three equalities relating the primal slack variables to the dual variables. If μ is set to zero in (2.4), then we see that the last three equations are exactly the complementarity conditions. Usualy, these last three equations are called $\mu-complementarity$ conditions. The KKT optimality conditions is a nonlinear algebraic system with $5n+2m+me$ equations and a similar number of unknowns, parametrized by the barrier parameter μ. The basis of the numerical algorithm for solving the nonlinear system (2.4) is the Newton's method, which is known to be very efficient for linear and convex quadratic programming. Assuming that the system (2.4) has a solution, then for each $\mu > 0$ we get a solution $(x_\mu, s_\mu, w_\mu, v_\mu, y_\mu, z_\mu, p_\mu, q_\mu)$. The path $\{(x_\mu, s_\mu, w_\mu, v_\mu, y_\mu, z_\mu, p_\mu, q_\mu) : \mu > 0\}$ is called the *primal-dual central path.* This path plays a fundamental role in interior-point methods for mathematical programming. The interior-point method which will be presented in the next sections is an iterative procedure that at each iteration attempts to move toward a point on the central path that is closer to the optimal point than the current point.

If the barrier parameter μ is set to zero, the KKT optimality conditions (2.4) can be partitioned in two classes as:

$$CO(t) = \begin{bmatrix} \nabla f(x) - \nabla g(x)^T y - \nabla h(x)^T z - p + q \\ g(x) \\ h(x) - s \\ x - w - l \\ x + v - u \end{bmatrix} \tag{2.5}$$

and

$$CT(s, z, w, p, v, q) = \begin{bmatrix} SZe \\ WPe \\ VQe \end{bmatrix} \tag{2.6}$$

The realtions (2.5) contain the primal and the dual constraints of the problem (1.1). On the other hand, the (2.6) relations contain the complementarity conditions. For this reason, the last three equations in (2.4) are called the $\mu-complementarity$ *conditions.* It is quite clear that if we could have a point $t = [x, y, z, p, q, s, w, v]$ satisfying the relation:

$$F(t) = \begin{bmatrix} CO(t) \\ CT(s, z, w, p, v, q) \end{bmatrix} = 0 \tag{2.7}$$

than the component x of this point is the solution of the problem (1.1). For solving the system (2.7) the Newton's method will be considered. Starting with an initial point, this implies the determination of a direction of moving and a step length along this direction. The next sections are dedicaed to this subjects.

The separation of the optimality conditions into these two classes is crucial for the elaboration of an efficient algorithm for (1.1), and for proving the convergence of this algorithm.

Remark. For linear programming problems the system (2.4) is much more simple. The only nonlinear expressions in (2.4) are simple multiplications of slack and dual variables, and the presence of these simple nonlinearity makes the subject of linear programming nontrivial. More than this, for linear programming case, if the primal and the dual feasible regions have nonempty interior, than there exists a critical point for the corresponding optimality conditions. (See, for example, Vanderebei [1996].) For the nonlinear programming case to prove the existence of a solution for the system (2.4) is more complicated.

3. The Newton System.

As above, denoting by $t = [x, y, z, p, q, s, w, v]^T$ than the Newton method applied to the system (2.7) consists of the determination of the *direction* $\Delta t = [\Delta x \ \Delta y \ \Delta z \ \Delta p \ \Delta q \ \Delta s \ \Delta w \ \Delta v]^T$ as solution of the following perturbed linear algebraic system:

$$F'(t)\Delta t = -F(t) + \mu \hat{e}, \tag{3.1}$$

where $\hat{e} \in R^{5n+2m+me}$ with zero components excepting the last $2n+m$ ones which are all equal with one, $F'(t)$ is the Jacobian of the function $F(t)$ computed in the current point t, and the computation of the new point

$$t^+ = t + \alpha \Delta t, \tag{3.2}$$

where the parameter α is the *size of the step length.*

Two problems are critical with this algorithm: the choice of α at each iteration and the possible modification of the system (3.1) in order to find a local minimizer of (1.1). For linear and convex quadratic programming , the modification of the linear system (3.1) is never required, excepting the situations of numerical difficulties, and the step length at each iteration is computed by a simple ratio test, [Lustig, Marsten and Shanno, 1990, 1991, 1992, 1994], [Vanderbei, 1990, 1994, 1996,], [Ye, 1997],[Wright, 1998]. For general convex nonlinear programming problems, it is well known that for a poor initial estimate the Newton's method may diverge, so again the linear system (3.1) need not be modified, but the technique for choosing the value of the parameter α at each iteration is more complex. In order to achieve convergence to a solution to the system (2.4), El-Bakry, Tapia, Tsuchiya and Zhang [1996] introduced a merit function and showed that for a proper choice of the barrier parameter μ, there exists an α at each iteration such that the algoritrhm (3.2) converge to a solution of (1.1), providing the Jacobian $F'(t)$ in (3.1) remains nonsingular. An implementation of this variant of the algorithm has been tested by Shanno, Breitfeld and Simantiraki [1996], and Shanno and Simantiraki [1997]. On the Hoch and Schittkowski [1981] set of test problems they found that while the algorithm was often efficient giving the optimal solution, it also often converged to local maxima or saddle points, all of which satisfying the first order necessary optimality conditions. More than this, for some problems, the Jacobian $F'(t)$ sometimes becomes singular, causing troubles to the algorithm. A modification of the merit function of El-Bakry *et al.* has been considered by Vanderbei and Shanno [1997] where the l_2 norm of the constraints multiplied by a parameter β is introduced. They prooved that there exists a value of β such that the direction given by the Newton system is a descent direction for this merit function. The corresponding algorithm has been implemented as a nonlinear version of LOQO [Vanderbei, 1994] and tested on the Hoch and Schittkowski set of problems, 8 large-scale nonlinear programming problems, and the Hans Mittelmann's quadratic programming set of problems, [Mittelmann. 1997]. Comparisons with MINOS [Murtagh and Saunders, 1995] and LANCELOT [Conn, Gould and Toint, 1992] showed how robust and efficient this interior point approach has proved to be.

4. The Newton Direction Determination.

The Newton system for the direction determination involves the computation of the Jacobian matrix of the function $F(t)$ in (2.7). From (2.4) we get the following linear algebraic system:

$$\begin{bmatrix} K(x,y,z) & -G(x)^T & -H(x)^T & -I & I & & & \\ G(x) & & & & & & & \\ H(x) & & & & & -I & & \\ I & & & & & & -I & \\ I & & & & & & & I \\ & & S & & & Z & & \\ & & & W & & & P & \\ & & & & V & & & Q \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta p \\ \Delta q \\ \Delta s \\ \Delta w \\ \Delta v \end{bmatrix} = \begin{bmatrix} \chi \\ -g(x) \\ -h(x)+s \\ l+w-x \\ u-v-x \\ -SZe+\mu e \\ -WPe+\mu e \\ -VQe+\mu e \end{bmatrix} \tag{4.1}$$

This system is not symmetric, but it can be very easy symmetrized by multiplying the second and the third equation by -1, and the last three equations by S^{-1}, W^{-1} and V^{-1} respectively. After this very simple algebraic operation, and eliminating the variables Δs, Δw and Δv as:

$$\begin{aligned}\Delta s &= \mu Z^{-1}e - Se - Z^{-1}S\Delta z,\\ \Delta w &= \mu P^{-1}e - We - P^{-1}W\Delta p,\\ \Delta v &= \mu Q^{-1}e - Ve - Q^{-1}V\Delta q,\end{aligned} \tag{4.2}$$

the following *reduced Newton system* is obtained:

$$\begin{bmatrix} K(x,y,z) & -G(x)^T & -H(x)^T & -I & I \\ -G(x) & & & & \\ -H(x) & & -Z^{-1}S & & \\ -I & & & -P^{-1}W & \\ I & & & & -Q^{-1}V \end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\\ \Delta z\\ \Delta p\\ \Delta q\end{bmatrix} = \begin{bmatrix}\chi\\ g(x)\\ h(x)-\mu Z^{-1}e\\ x-l-\mu P^{-1}e\\ u-x-\mu Q^{-1}e\end{bmatrix} \tag{4.3}$$

where:

$$K(x,y,z) = \nabla^2 f(x) - \sum_{i=1}^{me} y_i \nabla^2 g_i(x) - \sum_{i=1}^{m} z_i \nabla^2 h_i(x), \tag{4.4}$$

$$\begin{aligned} G(x) &= \nabla g(x),\\ H(x) &= \nabla h(x),\\ \chi &= -\nabla f(x) + G(x)^T y + H(x)^T z + p - q.\end{aligned} \tag{4.5}$$

Notice that the matrix of the reduced Newton system (4.3) is symmetric, undefined and $(3n+m+me)-$dimensional. Solving this system, or a reduction of it, together with (4.2) we get the direction $\Delta t = [\Delta x\ \Delta y\ \Delta z\ \Delta p\ \Delta q\ \Delta s\ \Delta w\ \Delta v]^T$. The existence of a solution for (4.3) implies the existence of the inverse of the diagonal matrices Z, P and Q,thus a condition of initialization of the algorithm is assumed to be satisfied.

The standard Newton's method assumptions for the problem (1.1) are as follows [Dennis and Schnabel, 1983]:

(A1) There exists a solution for the problem (1.1) as well as the associated dual variables, satisfying the KKT conditions given by (2.4).

(A2) The Hessian matrices $\nabla^2 f(x)$, $\nabla^2 g_i(x)$, $\nabla^2 h_i(x)$ for all i exist and are locally Lipschitz continuous at x^*.

(A3) The set of vectors $\{\nabla g_1(x^*), ..., \nabla g_{me}(x^*)\} \cup \{\nabla h_i(x^*) : i \in A(x^*)\}$ is linearly independent, where $A(x^*)$ is the set of inequality constraints active in x^*.

(A4) For every vector $d \neq 0$ satisfying $\nabla g_i(x^*)^T d = 0$, $i = 1, ..., me$ and $\nabla h_i(x^*)^T d = 0$, $i \in A(x^*)$ we have $d^T K(x^*, y^*, z^*) d > 0$.

(A5) For $i = 1, ..., m$, $z_i h_i(x^*) > 0$, and for $j = 1, ..., n$, $p_j(x_j^* - l_j) > 0$, $q_j^*(u_j - x_j^*) > 0$.

Proposition 1. *Supposing that the conditions (A1)-(A5) are satisfied and $s^* = h(x^*)$, $w^* = x^* - l$, $v^* = u - x^*$, than the Jacobian matrix $F'(t^*)$ of $F(t)$ given by the matrix from (4.1) is nonsingular.*

Proof. This follows immediately. Let the reduced problem be, in which only the inequality constraints active in the optim point are considered. Than, from the theory of equality constrained optimization, it follows that the matriceal block given by the first five rows and columns of the matrix from (4.1) is nonsingular. Hence, the nonsingularity of (4.1) follows from the strict complementarity condition (A5) and the nonsingularity of the above matriceal block from (4.1).□

Generally, in current implementations, the reduced Newton system (4.3) is not considered in form in which it is presented. Usually, we continue to reduce it by simple algebraic manipulations. Considering (4.3) and eliminating the variables Δz, Δp and Δq as:

$$\begin{aligned} \Delta z &= \mu S^{-1} e - S^{-1} Z h(x) - S^{-1} Z H(x) \Delta x, \\ \Delta p &= \mu W^{-1} e - W^{-1} P(x - l) - W^{-1} P \Delta x, \\ \Delta q &= \mu V^{-1} e - V^{-1} Q(u - x) + V^{-1} Q \Delta x, \end{aligned} \tag{4.6}$$

the system (4.3) is reduced at the following linear symmetric system:

$$\begin{bmatrix} \bar{K} & -G(x)^T \\ G(x) & \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} r + \mu \, \bar{r} \\ g(x) \end{bmatrix}, \tag{4.7}$$

where:

$$\begin{aligned} \bar{K} &= K(x, y, z) + H(x)^T S^{-1} Z H(x) + W^{-1} P + V^{-1} Q, \\ r &= \chi - H(x)^T S^{-1} Z h(x) - W^{-1} P(x - l) + V^{-1} Q(u - x), \\ \bar{r} &= H(x)^T S^{-1} e + W^{-1} e - V^{-1} e. \end{aligned} \tag{4.8}$$

Solving the system (4.7) we get the variables Δx and Δy with which from (4.6) and (4.2) the rest of variables of the system, (4.1) can be computed. Notice that, as in the case of linear programming, we must solve a symmetric, undefined, algebraic system, but in this case this is much more complicated, the Jacobian and Hessian matrices of the functions of the problems being involved.

5. The Merit Function.

Solving the system (4.7), and considering (4.6) and (4.2) respectively, we get a direction Δt pointing towards the central path. The problem we must solve now is to determine the step length along this direction. This is achieved by means of a merit function associated to the optimality conditions (2.4). The idea of a merit function is to ensure a progress toward a local minimum of the problem by conserving the feasibility of the solution. The merit function used for the line search is the squared l_2-norm of the KKT conditions (2.4), i.e.,

$$\Phi(t) = \parallel F(t) \parallel_2^2 = F(t)^T F(t). \tag{5.1}$$

Denote by $\Phi_k = \Phi(t_k)$ as the value of the merit function at the itration t_k, and $\Phi_k(\alpha) = \Phi(t_k + \alpha \Delta t_k)$ which illustrates the dependence of the merit function by the step length α. Clearly, $\Phi_k(0) = \Phi(t_k) = \Phi_k$. Hence,

$$\Phi_k(\alpha) = F(t_k + \alpha \Delta t_k)^T F(t_k + \alpha \Delta t_k). \tag{5.2}$$

The following proposition shows that the direction given by the perturbed Newton's system (3.1):

$$\Delta t_k = \left[F'(t_k)\right]^{-1} \left[-F(t_k) + \mu \hat{e}\right] \tag{5.3}$$

is a descent direction for the merit function $\Phi(t)$.

Proposition 2. *The direction Δt_k given by (5.3), solution of the perturbed Newton system (3.1) is a descent direction for the merit function (5.1).*

Proof. Considering the derivative of $\Phi_k(\alpha)$ at $\alpha = 0$ we get:

$$\begin{gathered}\Phi'(0) = 2F(t_k)^T F'(t_k) \Delta t_k = \\ 2F(t_k)^T F'(t_k) \left[F'(t_k)\right]^{-1} \left[-F(t_k) + \mu \hat{e}\right] = \\ 2F(t_k)^T \left[-F(t_k) + \mu \hat{e}\right] = \\ -2 \parallel F(t_k) \parallel_2^2 + 2\mu F(t_k)^T \hat{e},\end{gathered}$$

hence, $\Phi'(0) < 0$ if and only if $2\mu F(t_k)^T \hat{e} \leq 2 \parallel F(t_k) \parallel_2^2$. This suggests the following estimation of the barrier parameter used into the logarithmic barrier function (2.2):

$$\mu \leq \frac{\|F(t_k)\|_2^2}{F(t_k)^T \hat{e}} = \frac{\|F(t_k)\|_2^2}{s^T z + w^T p + v^T q}. \tag{5.4}$$

So, at every iteration, choosing the value of the barrier parameter μ as in (5.4), the direction Δt_k given by (5.3) is indeed a descent direction for the merit function $\Phi(t_k)$.□

The realation (5.4) is a very important estimation of the barrier parameter μ. We will see that the numerical experiments recommend a much smaller value for it.

6. The Step Length Determination.

The step length computation in algorithm (3.2) is a very important ingredient. For the unconstrained optimization the value of α is determined by means of the *Wolfe conditions*, [Wolfe, 1969, 1971], [Zoutendijk, 1970], [Dennis and Schnabel, 1983] , [Nocedal, 1992]. For linear programming or linear constrained optimization the step length is determined by a simple ratio test. For nonlinear programming the strategy for choosing α at each iteration becomes more complex. In this case it is necessary to impose some additional constraints to ensure that the current iterations are generated as close as possible to the central path.

As we have learned from linear programming, also for problem (1.1), the distance from centrality is given by:

$$\xi_k = \frac{\min\{s_i z_i,\ w_i p_i,\ v_i q_i\}}{\frac{s^T z + w^T p + v^T q}{2n+m}}. \tag{6.1}$$

Clearly, $0 < \xi_k \leq 1$, and $\xi_k = 1$ if and only if $s_i z_i$, $w_i p_i$, $v_i q_i$ are equal with a constant for all values of i. To specity a value of α, firstly the following function is introduced:

$$\Theta^I(\alpha) = \frac{\min\{s_i(\alpha)z_i(\alpha),\ w_i(\alpha)p_i(\alpha),\ v_i(\alpha)q_i(\alpha)\}}{\frac{s^T(\alpha)z(\alpha)+w^T(\alpha)p(\alpha)+v^T(\alpha)q(\alpha)}{2n+m}} - \gamma\tau_1 \tag{6.2}$$

where τ_1 is the initial distance from centrality:

$$\tau_1 = \frac{\min\{s_i^0 z_i^0,\ w_i^0 p_i^0,\ \mathrm{v}_i^0 q_i^0\}}{\frac{s^{0T} z^0 + w^{0T} p^0 + v^{0T} q^0}{2n+m}}, \tag{6.3}$$

and $\gamma \in (0,1)$ is a constant with which we can modify the distance from centrality. Notice that for $t = t_0$ and $\gamma = 1$, it follows that $\Theta^I(0) = 0$. More than this, $\Theta^I(\alpha)$ is a piecewise quadratic function. To choose the step length α_k at each iteration we will require that α_k to satisfy $\Theta^I(\alpha) \geq 0$ for all $\alpha \in [0, \alpha_k]$, i.e.

$$\frac{\min\{s_i(\alpha)z_i(\alpha),\ w_i(\alpha)p_i(\alpha),\ v_i(\alpha)q_i(\alpha)\}}{\frac{s^T(\alpha)z(\alpha)+w^T(\alpha)p(\alpha)+v^T(\alpha)q(\alpha)}{2n+m}} \geq \gamma_k\tau_1 \tag{6.4}$$

where the variables of the problem are considered at the $k-$th iteration and $\gamma_k \in (0,1)$. Since $\Theta^I(\alpha)$ is a piecewise quadratic function, from (6.4) it follows that α_k can be easily computed.

Considering now the merit function (5.1), the Wolfe's conditions for its minimization are:

$$\Phi(t_{k+1}) \leq \Phi(t_k) + \beta\alpha_k \nabla\Phi(t_k)^T \Delta t_k, \tag{6.5}$$

$$\nabla\Phi(t_{k+1})^T \Delta t_k \geq \delta\nabla\Phi(t_k)^T \Delta t_k, \tag{6.6}$$

where $0 < \beta < \delta < 1$ are parameters fixing the reduction of the merit function, as well as the rate of decreasing of this function along the direction Δt_k. Having in view that $\Phi(t_{k+1}) = \Phi_k(\alpha)$, the first Wolfe's condition (6.5) is equivalent to:

$$\Phi_k(\alpha) \leq \Phi_k(0) + \beta\alpha_k \Phi_k'(0). \tag{6.7}$$

Proposition 3. *For the merit function (5.1), taking*

$$\mu_k = \sigma_k \frac{s_k^T z_k + w_k^T p_k + v_k^T q_k}{2n+m} \tag{6.8}$$

where $\sigma_k \in (0,1)$ we have:

$$\Phi_k'(0) = -2\left[\Phi_k(0) - \frac{\sigma}{2n+m}(s^T z + w^T p + v^T q)^2\right]. \tag{6.9}$$

Proof. By direct computation we get:

$$\Phi_k'(0) = -2\Phi_k(0) + 2\mu F(t_k)^T \hat{e} = -2\Phi_k(0) + 2\mu[s^T z + w^T p + v^T q] =$$
$$-2\Phi_k(0) + 2\sigma\frac{s^T z + w^T p + v^T q}{2n+m}[s^T z + w^T p + v^T q] =$$
$$-2\left[\Phi_k(0) - \frac{\sigma}{2n+m}(s^T z + w^T p + v^T q)^2\right]. \square$$

Proposition 4. *At every iteration:*

$$\frac{(s^T z + w^T p + v^T q)^2}{2n+m} \leq \Phi_k(0). \tag{6.10}$$

Proof. By simple algebraic manipulation we get:

$$\frac{(s^T z + w^T p + v^T q)^2}{2n+m} \leq \| SZe \|_2^2 + \| WPe \|_2^2 + \| VQe \|_2^2 \leq$$
$$\| SZe \|_2^2 + \| WPe \|_2^2 + \| VQe \|_2^2 + \| CO(t_k) \|_2^2 = \Phi_k(0). \square$$

It is easy to show that the estimation of the barrier parameter μ_k given by (6.8) with $\sigma_k \in (0,1)$, is smaller than the estimation recommended by (5.4). Hence, the μ_k given by (6.8) ensures the descent character of the direction Δt_k. Moreover, the following proposition gives an estimation of the reduction of the values of the merit function.

Proposition 5. *The direction Δt_k solution of the perturbed Newton system (3.1) with μ given by (6.8) is a descent direction for the merit function $\Phi(t)$ at every t_k. Moreover, if the first Wolfe condition (6.5) is satisfied, then*

$$\Phi_k(\alpha_k) \leq [1 - 2\alpha_k\beta(1-\sigma_k)]\Phi_k(0). \tag{6.11}$$

Proof. As we know $\Phi'_k(0) = -2 \parallel F(t_k) \parallel_2^2 + 2\mu_k(s^Tz + w^Tp + v^Tq)$. Considering μ_k as in (6.8) it follows that:

$$\begin{gathered}\Phi'_k(0) = -2\Phi_k(0) + 2\sigma_k\tfrac{(s^Tz+w^Tp+v^Tq)^2}{2n+m} \leq \\ -2\Phi_k(0) + 2\sigma_k\Phi_k(0) = -2\Phi_k(0)(1-\sigma_k) < 0,\end{gathered}$$

prooving the descent character of the direction given by the perturbed Newton system. Moreover, from (6.7) and taking into account the above propositions 3 and 4 we have:

$$\begin{gathered}\Phi_k(\alpha) \leq \Phi_k(0) + \beta\alpha_k\Phi'_k(0) = \\ \Phi_k(0) + \beta\alpha_k\left[-2\left(\Phi_k(0) - \sigma_k\tfrac{(s^Tz+w^Tp+v^Tq)^2}{2n+m}\right)\right] \leq \\ \Phi_k(0) - 2\beta\alpha_k\Phi_k(0) + 2\beta\alpha_k\sigma_k\Phi_k(0) = \\ \left[1 - 2\alpha_k\beta\left(1-\sigma_k\right)\right]\Phi_k(0),\end{gathered}$$

which proves the proposition. □

This proposition shows that the sequence $\{\Phi_k\}$ is monotone and nonincreasing, therefore for all $k : \Phi_k \leq \Phi_0$. Moreover, if the sequence of step length $\{\alpha_k\}$ is bounded away from zero and the parameter σ_k is bounded away from one at every iteration, it follows that the merit function is linear convergent to zero. Hence, the above (6.11) inequality is equivalent to:

$$\tfrac{\|F(t_{k+1})\|_2}{\|F(t_k)\|_2} \leq [1 - 2\alpha_k\beta(1-\sigma_k)]^{1/2}. \tag{6.12}$$

Some numerical examples illustrate that a problem that may lead to the nonconvergence of the algorithm is that the sequence $\{\parallel CT(s_k, z_k, w_k, p_k, v_k, q_k) \parallel \}$ converges to zero faster than the sequence $\{\Phi(t_k)\}$. In these cases the sequence of the steplengths $\{\alpha_k\}$ is decreasing to zero, thus determining the nonconvergence of the algorithm. In order to avoid this situation let us introduce the following function:

$$\Theta^{II}(\alpha) = s(\alpha)^T z(\alpha) + w(\alpha)^T p(\alpha) + v(\alpha)^T q(\alpha) - \gamma\tau_2 \parallel CO(t(\alpha)) \parallel_2 \tag{6.13}$$

where

$$\tau_2 = \frac{s_0^T z_0 + w_0^T p_0 + v_0^T q_0}{\|CO(t_0)\|_2}, \tag{6.14}$$

and $\gamma \in (0,1)$ is a constant, the same as in (6.2). Notice that for $t = t_0$ and $\gamma = 1$; $\Theta^{II}(0) = 0$. $\Theta^{II}(\alpha)$ is generally nonlinear. For choosing the steplength α_k at every iteration, we will require that α_k to satisfy :

$$\Theta^{II}(\alpha_k) \geq 0. \tag{6.15}$$

Proposition 6. *Let $\{t_k\}$ be a sequence generated by (3.1). Then*

$$\min\{1, 0.5\tau_2\}\Phi(t_k) \leq (s_k^T z_k + w_k^T p_k + v_k^T p_k)^2 \leq (2n+m)\Phi(t_k). \tag{6.16}$$

Proof. The second inequality folows from proposition 4. So, we will need to prove only the first one. Since $\Theta^i(\alpha_k) \geq 0$, for $i = 1, 2$, from (6.13) with $\gamma_k \geq 1/2$, we have:

$$(s_k^T z_k + w_k^T p_k + v_k^T p_k) \geq (1/2)\tau_2 \parallel CO(t_k) \parallel_2 .$$

It follows that

$$\begin{gathered}(s_k^T z_k + w_k^T p_k + v_k^T p_k) \geq \\ \tfrac{1}{2}[\parallel SZe \parallel_2 + \parallel WPe \parallel_2 + \parallel VQe \parallel_2 + 0.5\tau_2 \parallel CO(t_k) \parallel_2] \geq \\ \tfrac{1}{2}\min\{1, 0.5\tau_2\} \parallel F(t_k) \parallel_2,\end{gathered}$$

which completes the proof. □

This proposition shows that the transversality conditions are bounded, and γ_k must be selected as a decreasing sequence with $1/2 \leq \gamma_k \leq \gamma_{k-1}$. Having in view all these developments, at each iteration, the steplength α_k is computed as a solution of the following system of algebraic inequalities:

$$
\begin{array}{ll}
a) & l \leq x_k + \alpha_k \Delta x_k \leq u, \\
b) & s(\alpha_k), z(\alpha_k), w(\alpha_k), p(\alpha_k), v(\alpha_k), q(\alpha_k) > 0, \\
c) & \Theta^I(\alpha) \geq 0, \alpha \in [0, \alpha_k], \\
d) & \Theta^{II}(\alpha_k) \geq 0, \\
e) & \Phi_k(\alpha_k) \leq \Phi_k(0) + \alpha_k \beta \Phi_k'(0), \\
f) & \nabla \Phi(\alpha_k)^T \Delta t_k \geq \delta \nabla \Phi(0)^T \Delta t_k.
\end{array}
\tag{6.17}
$$

where $0 < \beta < \delta < 1$.

In order to determine a value α_k for the steplength satisfying (6.17) we shall consider a strategy of *interval reducing*. Clearly, the first two relations (6,17a) and (6,17b) are very simple to be implemented. Exactly as in the linear programming case the coresponding ratio test is performed thus obtaining a value α_m which maintain the positivity of the variables as well as simple bounds on the variables of the problem. Then, a value $\alpha_k \in (0,\ \alpha_m]$ which satisfy the conditions (6.17c - 6.17f) is selected.

7. The Primal-Dual Interior-Point Algorithm

The interior point algorithm for solving (1.1), based on the above developments, has three main parts which refer to the computing of: the direction of moving, the barrier parameter, and the steplength.

Algorithm PCNC

Step 1. *Initialization.* Choose the initial point $t_0 = [x_0\ y_0\ z_0\ p_0\ q_0\ s_0\ w_0\ v_0]$ such that $l \leq x_0 \leq u$, $(s_0, z_0) > 0$, $(w_0, p_0) > 0$, $(v_0, q_0) > 0$, as well as the values of the parameters: $\beta \in (0, 1/2]$, $\gamma_{k-1} = 1$, and $\rho \in (0, 1)$. Set $k = 0$.

Step 2. *Test of convergence.* Compute the value of the merit function: $\Phi(t_k) = F(t_k)^T F(t_k)$. If $\Phi(t_k) \leq \varepsilon$, stop, else continue with step 3.

Step 3. *Evaluation of the barier parameter.* Choose $\sigma_k \in (0, 1)$. Using (6.8) compute the value of the barrier parameter μ_k.

Step 4. *Computing the direction of moving.* Compute the perturbed Newton direction Δt_k by solving the reduced Newton system (4.7) and using the relations (4.6) and (4.2).

Step 5. *Steplength determination.*

a) Choose $\frac{1}{2} \leq \gamma_k \leq \gamma_{k-1}$.

b) Compute α_m as the maximum value of α_k which satisfy the conditions (6.17a) and (6.17b).

c) Determine $\alpha_k^I \in (0, \alpha_m]$ as the smallest positive root such that $\Theta^I(\alpha) \geq 0$ for all $\alpha \in (0, \alpha_k^I]$.

d) Determine $\alpha_k^{II} \in (0, \alpha_k^I]$ such that $\Theta^{II}(\alpha_k^{II}) \geq 0$.

e) Set $\bar{\alpha}_k = \min(\alpha_k^I, \alpha_k^{II})$.

f) Let $\alpha_k = \rho^j \ \bar{\alpha}_k$, where j is the smallest nonnegative integer such that α_k satisfies the condition (6.17e)

Step 6. *Updating of the variables.* Set $t_{k+1} = t_k + \alpha_k \Delta t_k$, $k = k+1$ and go to step 2.

Some remarks are in order:

1. *The initial point.* Generally, nonlinear programming requires that an initial starting point be given as part of the problem data. For an interior-point algorithm besides the specification of the initial values for x_j, the initial values for the variables y_i, z_i, p_i, etc. must be determined by the program. For slack variables w_i and v_i we can consider simply the initial values as: $w_i = x_i - l_i$ and $v_i = u_i - x_i$, for $i = 1, ..., n$, respectively. For slack variables s_i we can compute $h(x_0)$ and set $s_i = h_i(x_0)$, for $i = 1, ..., m$. There are two problems with this approach for computing the initial values for the s_i variables. Firstly, if x_0 is not feasible, than s_i computed as above gives an initial negative value to some components. Secondly, even if x_0 is feasible, it may lie on the boundary of the feasible region of the problem or so close to it that some initial values of s_i are very close to zero precluding the algorithm to make any progress. To overcome this difficulty, as Vanderbei and Shanno [1997] suggested, it is necessary to specify a value for a parameter $\vartheta > 0$ so that all initial values of variables constrained to be nonnegative are at least as large as ϑ. Hence s_i is initially set as: $s_i = \max\{h_i(x_0), \vartheta\}$. (In our tests we have considered $\vartheta = 1$.) The initial values of the variables y_i, $i = 1, ..., me$, are computed as solution of the following linear system:

$$[\nabla g(x_0) \nabla g(x_0)^T] y = \nabla g(x_0)[\nabla f(x_0) - \nabla h(x_0)^T z_0 - p_0 + q_0]. \qquad (7.1)$$

Proposition 7. *Suppose that the functions of the problem $f(x)$, $g(x)$ and $h(x)$ are twice continuously differentiable and the derivative of $CO(t)$ given by (2.5) is Lipschitz continuous. If the columns of $\nabla g(x)$ are linearly independent, than the variable y given by (7.1) is well defined and bounded.*

Proof. From the first relation of (2.5) it follows that:

$$y = [\nabla g(x) \nabla g(x)^T]^{-1} \nabla g(x)[\nabla f(x) - \nabla h(x)^T z - p + q]. \qquad (7.2)$$

Having in view that x is in a compact set and the Jacobian $\nabla g(x)$ is of full rank on rows, than the inverse of the matrix in (7.2) exists, i.e. y is well defined. The boundness of y follows from the conditions of the proposition. □

2. *The search direction.* In proposition 2 above we showed that the step

direction Δt given by (5.3) have desirable descent properties for the merit function provided that the value of the barrier parameter is selected as in (5.4).

To determine the search direction the reduced Newton system (4.7) is solved by Cholesky factorization. After Δx and Δy have been determined using (4.6) and (4.2) the rest of variables are very easy computed.

Proposition 8. *Suppose that the functions of the problem $f(x)$, $g(x)$ and $h(x)$ are twice continously differentiable, the derivative of $CO(t)$ given by (2.5) is Lipschitz continuous and the set of gradients $\{\nabla g_1(x_k), ..., \nabla g_{me}(x_k)\} \cup \{\nabla h_i(x_k),$ $i \in A(x^*)\}$ is linearly independent for k sufficiently large, where $A(x^*)$ is the set of active inequality constraints in x^*. Then the sequence of search directions $\{\Delta t_k\}$ generated by the algorithm PCNC is bounded.*

Proof. Symmetrizing the matrix $F'(t_k)$ and rearranging the order of its rows and columns, we have

$$F'(t_k) = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

where

$$A = \begin{bmatrix} K(x_k, y_k, z_k) & -G(x_k)^T & -H(x_k)^T \\ -G(x_k) & & \\ -H(x_k) & & \end{bmatrix}$$

and the matrices B and C can be very easy identified from $F'(t)$. By assumptions of the proposition the matrix A is invertible and $\| A^{-1} \|$ is uniformly bounded. A straightforward computation shows that:

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} =$$
$$\begin{bmatrix} A^{-1} + A^{-1}B(C - B^TA^{-1}B)^{-1}B^TA^{-1} & -A^{-1}B(C - B^TA^{-1}B)^{-1} \\ -(C - B^TA^{-1}B)^{-1}B^TA^{-1} & (C - B^TA^{-1}B)^{-1} \end{bmatrix}$$

which is bounded, since every matrix involved is bounded. Hence, $\left[F'(t_k)\right]^{-1}$ is uniformly bounded, which proves the proposition. □

3. *Measures of progress.* We assume that the functions of the problem $f(x)$, $g(x)$ and $h(x)$ are twice continuously differentiable and the derivative of $CO(t)$ in (2.5) is Lipschitz continuous. From (6.17a) we notice that the iteration sequence $\{x_k\}$ is bounded. More than this, the descent character of the Newton direction and the specific choice of the steplength α ensure that $\Delta t \to 0$ along the iterations. The following propositions show the progress of the algorithm in one iteration regarding the primal and the dual feasibility as well as the complementarity.

Proposition 9. *Let us define:*

$$\begin{aligned} e_1(x,s) &= s - h(x), \\ e_2(x,w) &= l + w - x, \\ e_3(x,v) &= u - v - x. \end{aligned} \tag{7.3}$$

Then

$$\begin{aligned} e_1(\bar{x},\bar{s}) &= (1-\alpha)e_1(x,s) + o(\alpha) \\ e_2(\bar{x},\bar{w}) &= (1-\alpha)e_2(x,w) \\ e_3(\bar{x},\bar{v}) &= (1-\alpha)e_3(x,v) \end{aligned} \tag{7.4}$$

Proof. By direct computation we have:

$$\begin{gathered} e_1(\bar{x},\bar{s}) = \bar{s} - h(\bar{x}) = s + \alpha\Delta s - h(x+\alpha\Delta x) = \\ s + \alpha\Delta s - h(x) - \alpha\nabla h(x)\Delta x + o(\alpha) = \\ s - h(x) + \alpha[\Delta s - \nabla h(x)\Delta x] + o(\alpha) = \\ (1-\alpha)e_1(x,s) + o(\alpha), \end{gathered}$$

where the last equality follows from the third block of equations in (4.1). The analysis is similar for e_2 and e_3.□

Proposition 10. *Define* $\bar{\chi}$ *as the value of* χ *from (4.5) computed in* $\bar{t} = t + \alpha\Delta t$*. Then*

$$\bar{\chi} \leq (1-\alpha)\chi. \tag{7.5}$$

Proof. By direct computation we get:

$$\begin{gathered} \bar{\chi} = -\nabla f(\bar{x}) + G(\bar{x})^T\bar{y} + H(\bar{x})^T\bar{z} + \bar{p} - \bar{q} = \\ \chi + \alpha[-K(x,y,z)\Delta x + G(x)^T\Delta y + H(x)^T\Delta z + \Delta p - \Delta q] + \\ \alpha^2[\sum_{i=1}^{me}\Delta y_i\nabla^2 g_i(x) + \sum_{i=1}^{m}\Delta z_i\nabla^2 h_i(x)]\Delta x. \end{gathered}$$

Using the first equality from (4.1) it follows that

$$\bar{\chi} = (1-\alpha)\chi + \alpha^2[\sum_{i=1}^{me}\Delta y_i\nabla^2 g_i(x) + \sum_{i=1}^{m}\Delta z_i\nabla^2 h_i(x)]\Delta x.$$

Now having in view that the derivative of $CO(t)$ from (2.5) is Lipschitz continuous it follows that $\bar{\chi} \leq (1-\alpha)\chi$. $\square$

Proposition 11. *Assume that* $\| s \|_\infty, \| w \|_\infty, \| v \|_\infty$ *are bounded by a large real number* M, *and* α *defined by (6.17) is such that* $\alpha \leq 1$. *Let us define:*

$$\omega = s^T z + w^T p + v^T q. \tag{7.6}$$

Then

$$\bar{\omega} \leq \omega[1-\alpha(1-\sigma)] + M[\| \Delta z \|_1 + \| \Delta p \|_1 + \| \Delta q \|_1] \tag{7.7}$$

Proof. By simple algebraic manipulations we get:

$$\begin{gathered}
\bar{\omega} = \bar{s}^T \bar{z} + \bar{w}^T \bar{p} + \bar{v}^T \bar{q} = \\
(s+\alpha\Delta s)^T(z+\alpha\Delta z) + (w+\alpha\Delta w)^T(p+\alpha\Delta p) + (v+\alpha\Delta v)^T(q+\alpha\Delta q) = \\
\omega + \alpha[(s^T\Delta z + z^T\Delta s) + (w^T\Delta p + p^T\Delta w) + (v^T\Delta q + q^T\Delta v)] + \\
\alpha^2[\Delta s^T\Delta z + \Delta w^T\Delta p + \Delta v^T\Delta q].
\end{gathered}$$

But,

$$\begin{gathered}
s^T\Delta z + z^T\Delta s = e^T(S\Delta z + Z\Delta s) = e^T(\mu e - SZe) = \mu m - s^T z, \\
w^T\Delta p + p^T\Delta w = e^T(W\Delta p + P\Delta w) = e^T(\mu e - WPe) = \mu n - w^T p, \\
v^T\Delta q + q^T\Delta v = e^T(V\Delta q + Q\Delta v) = e^T(\mu e - VQe) = \mu n - v^T q,
\end{gathered}$$

where the last two equalities are from (4.1). So,

$$\bar{\omega} = \omega + \alpha[(m+2n)\mu - \omega] + \alpha^2[\Delta s^T\Delta z + \Delta w^T\Delta p + \Delta v^T\Delta q],$$

and from (6.8) it follows that:

$$\bar{\omega} = \omega[1-\alpha(1-\sigma)] + \alpha^2[\Delta s^T\Delta z + \Delta w^T\Delta p + \Delta v^T\Delta q].$$

Now we shall estimate the term in α^2 as follows:

$$\alpha^2 \mid \Delta s^T\Delta z + \Delta w^T\Delta p + \Delta v^T\Delta q \mid \leq \alpha^2[\mid \Delta s^T\Delta z \mid + \mid \Delta w^T\Delta p \mid + \mid \Delta v^T\Delta q \mid],$$

But,

$$\begin{gathered}
\alpha^2 \mid \Delta s^T\Delta z \mid = \alpha^2 \mid \sum_i \Delta s_i \Delta z_i \mid \leq \alpha^2 \sum_i \mid \Delta s_i \mid\mid \Delta z_i \mid \leq \\
\alpha^2(\max_i \mid \Delta s_i \mid)(\sum_i \mid \Delta z_i \mid) = \alpha^2 \| \Delta s \|_\infty \| \Delta z \|_1 = \\
\alpha \| \alpha\Delta s \|_\infty \| \Delta z \|_1,
\end{gathered}$$

where the last inequality is a trivial case of the Hölder's inequality. Clearly, similar estimates can be obtained for $\alpha^2 \mid \Delta w^T \Delta p \mid$ and $\alpha^2 \mid \Delta v^T \Delta q \mid$. Substituting these expressions into the last expression for $\bar{\omega}$, we get:

$$\bar{\omega} \leq \omega[1-\alpha(1-\sigma)]+$$
$$\alpha[\| \alpha \Delta s \|_\infty \| \Delta z \|_1 + \| \alpha \Delta w \|_\infty \| \Delta p \|_1 + \| \alpha \Delta v \|_\infty \| \Delta q \|_1].$$

Now, we use the specific choice of the steplength α to get a bound on $\| \alpha \Delta s \|_\infty$, $\| \alpha \Delta w \|_\infty$ and $\| \alpha \Delta v \|_\infty$. Indeed, (6.17a and b) implies that

$$\alpha \leq \tfrac{s_i}{|\Delta s_i|}, \text{ for all } i = 1, ..., m, \text{ hence, } \| \alpha \Delta s \|_\infty \leq \| s \|_\infty .$$

Similarly, $\| \alpha \Delta w \|_\infty \leq \| w \|_\infty$ and $\| \alpha \Delta v \|_\infty \leq \| v \|_\infty$. Having in view that $\| s \|_\infty, \| w \|_\infty$ and $\| v \|_\infty$ are bounded by the real number M, the following estimate of the complementarity is obtained:

$$\bar{\omega} \leq \omega[1-\alpha(1-\sigma)] + M[\| \Delta z \|_1 + \| \Delta p \|_1 + \| \Delta q \|_1]. \ \Box$$

Another estimation of the complementarity can be given as in

Proposition 12. *Assume that $\| z \|_\infty, \| p \|_\infty, \| q \|_\infty$ are bounded by a large real number M, and α defined by (6.17) is such that $\alpha \leq 1$. Then*

$$\bar{\omega} = \leq \omega[1-\alpha(1-\sigma)] + M[\| e_1 \|_1 + \| e_2 \|_1 + \| e_3 \|_1]. \tag{7.8}$$

Proof. As in proposition 11 above we can write:

$$\bar{\omega} = \omega[1-\alpha(1-\sigma)] + \alpha^2[\Delta s^T \Delta z + \Delta w^T \Delta p + \Delta v^T \Delta q].$$

Now, from (4.1) we get: $\Delta s = H(x)\Delta x - e_1$, $\Delta w = \Delta x - e_2$ and $\Delta v = e_3 - \Delta x$. Substituting these expressions of Δs, Δw and Δv into the expresion of $\bar{\omega}$ we get

$$\bar{\omega} = \omega[1-\alpha(1-\sigma)] + \alpha^2[\Delta x^T(H^T \Delta z + \Delta p - \Delta q) - e_1^T \Delta z - e_2^T \Delta p + e_3^T \Delta q].$$

But from the first equality in (4.1) $H^T \Delta z + \Delta p - \Delta q = K \Delta x - G^T \Delta y - \chi$. Hence

$$\bar{\omega} = \omega[1-\alpha(1-\sigma)] + \alpha^2[\Delta x^T K \Delta x - \Delta x^T G^T \Delta y - \Delta x^T \chi - e_1^T \Delta z - e_2^T \Delta p + e_3^T \Delta q].$$

Having in view that the functions of the problem are twice continuously differentiable and the derivative of $CO(t)$ given by (2.5) is Lipschitz continuos, it follows that

$$\bar{\omega} \leq \omega[1-\alpha(1-\sigma)] + \alpha^2[\mid e_1^T \Delta z \mid + \mid e_2^T \Delta p \mid + \mid e_3^T \Delta q \mid].$$

As above, using a variant of the Hölder inequality we obtain

$$\bar{\omega} \leq \omega[1-\alpha(1-\sigma)] + \alpha[\| e_1 \|_1 \| \alpha\Delta z \|_\infty + \| e_2 \|_1 \| \alpha\Delta p \|_\infty + \| e_3 \|_1 \| \alpha\Delta q \|_\infty].$$

But, from the specific choice of α we can write: $\| \alpha\Delta z \|_\infty \leq \| z \|_\infty \leq M$, $\| \alpha\Delta p \|_\infty \leq \| p \|_\infty \leq M$ and $\| \alpha\Delta q \|_\infty \leq \| q \|_\infty \leq M$ which introduced into the last inequality we can estimate the new complementarity as:

$$\bar{\omega} \leq \omega[1-\alpha(1-\sigma)] + M[\| e_1 \|_1 + \| e_2 \|_1 + \| e_3 \|_1]. \ \square$$

As we know from the duality theory, three criteria must be met in order that a primal-dual solution be optimal: primal feasibility, dual feasibility and complementarity. The following theorems show the progress of the algorithm over several iterations with respect to the primal and dual feasability, as well as the complementarity.

Theorem 1. *Let $e_1^k(x,s)$, $e_2^k(x,w)$ and $e_3^k(x,v)$ denote the values of these quantities at the k-th iteration. Then:*

$$\begin{aligned} e_1^k(x,s) &\leq (1-\alpha)^k e_1^0(x,s), \\ e_2^k(x,w) &= (1-\alpha)^k e_2^0(x,w), \\ e_3^k(x,v) &= (1-\alpha)^k e_3^0(x,v). \end{aligned} \tag{7.9}$$

Proof. From proposition 9 we have $e_1^k(x,s) \leq (1-\alpha)e_1^{k-1}(x,s) \leq \ldots \leq (1-\alpha)^k e_1^0(x,s)$, which proves the first inequality of the proposition. Similarly, $e_2^k(x,w) = (1-\alpha)e_2^{k-1}(x,w) = \ldots\ (1-\alpha)^k e_2^0(x,w)$, and $e_3^k(x,v) = (1-\alpha)e_3^{k-1}(x,v) = \ldots = (1-\alpha)^k e_3^0(x,v)$. $\square$

Theorem 2. *Let χ^k denote the value of χ at the k-th iteration. Then:*

$$\chi^k \leq (1-\alpha)^k \chi^0. \tag{7.10}$$

Proof. From proposition 10 above we can write: $\chi^k \leq (1-\alpha)\chi^{k-1} \leq \ldots \leq (1-\alpha)^k \chi^0$, which proves the theorem. $\square$

Theorem 3. *Let ω^k denote the value of ω at the k-th iteration. Assume that $\| z \|_\infty$, $\| p \|_\infty$, $\| q \|_\infty$ are bounded by a large real number $M < \infty$, and α defined by (6.17) is such that $\alpha \leq 1$. Then there exists a constant $\widetilde{M} < \infty$ such that:*

$$\omega^k \leq (1-\bar{\alpha})^k \tilde{M} \tag{7.11}$$

where $\bar{\alpha}=\alpha(1-\sigma)$.

Proof. From proposition 12 and the previous estimations given by theorem 1, we see that

$$\begin{gathered}\omega^k \leq (1-\alpha(1-\sigma))\omega^{k-1} + M(1-\alpha)^{k-1}[\| e_1 \|_1 + \| e_2 \|_1 + \| e_3 \|_1] = \\ (1-\bar{\alpha})\omega^{k-1} + \bar{M}(1-\alpha)^{k-1},\end{gathered}$$

where $\bar{\alpha}=\alpha(1-\sigma)$ and $\bar{M}=M[\| e_1 \|_1 + \| e_2 \|_1 + \| e_3 \|_1]$. Since an analogous inequality relates ω^{k-1} to ω^{k-2}, we can substitute the corresponding inequality to get

$$\begin{gathered}\omega^k \leq (1-\bar{\alpha})[(1-\bar{\alpha})\omega^{k-2} + \bar{M}(1-\alpha)^{k-2}] + \bar{M}(1-\alpha)^{k-1} = \\ (1-\bar{\alpha})^2\omega^{k-2} + \bar{M}(1-\alpha)^{k-1}\left[\tfrac{1-\bar{\alpha}}{1-\alpha}+1\right].\end{gathered}$$

Continuing in this manner, we can write

$$\begin{gathered}\omega^k \leq (1-\bar{\alpha})^2[(1-\bar{\alpha})\omega^{k-3} + \bar{M}(1-\alpha)^{k-3}] + \bar{M}(1-\alpha)^{k-1}\left[\tfrac{1-\bar{\alpha}}{1-\alpha}+1\right] = \\ (1-\bar{\alpha})^3\omega^{k-3} + \bar{M}(1-\alpha)^{k-1}\left[\left(\tfrac{1-\bar{\alpha}}{1-\alpha}\right)^2 + \tfrac{1-\bar{\alpha}}{1-\alpha}+1\right] \leq \ldots \leq \\ (1-\bar{\alpha})^k\omega^{0} + \bar{M}(1-\alpha)^{k-1}\left[\left(\tfrac{1-\bar{\alpha}}{1-\alpha}\right)^{k-1} + \ldots + \tfrac{1-\bar{\alpha}}{1-\alpha}+1\right].\end{gathered}$$

But

$$\begin{gathered}(1-\alpha)^{k-1}\left[\left(\tfrac{1-\bar{\alpha}}{1-\alpha}\right)^{k-1} + \ldots + \tfrac{1-\bar{\alpha}}{1-\alpha}+1\right] = \\ \frac{\left(1-\bar{\alpha}\right)^k-(1-\alpha)^k}{\alpha-\bar{\alpha}} = \frac{\left(1-\bar{\alpha}\right)^k-(1-\alpha)^k}{\alpha\sigma} \leq \frac{\left(1-\bar{\alpha}\right)^k}{\alpha\sigma}.\end{gathered}$$

Considering this last inequality , we see that

$$\omega^k \leq (1-\bar{\alpha})^k\omega^0 + \bar{M}\,\frac{\left(1-\bar{\alpha}\right)^k}{\alpha\sigma} = (1-\bar{\alpha})^k\left[\omega^0 + \tfrac{\bar{M}}{\alpha\sigma}\right].$$

Denoting $\left[\omega^0 + \frac{\bar{M}}{\alpha\sigma}\right] = \tilde{M}$, we see that $\omega^k \leq (1 - \bar{\alpha})^k \; \tilde{M}$. $\square$

The above theorems are only a partial convergence result because all of them depend on the assumption that the steplengths remain bounded away from zero.

Remark. Note that the primal and dual infeasibilities go down by a factor of $(1-\alpha)$ at each iteration, whereas the complementarity goes down by the factor $(1- \bar{\alpha}) > (1-\alpha)$.

4. *The step length.* A crucial point of the algorithm is the steplength determination. The following proposition shows that the sequence of $\{\bar{\alpha}_k\}$ generated by step 5e of the algorithm PCNC is bounded away from zero.

Proposition 13. *Suppose that the functions of the problem $f(x)$, $g(x)$ and $h(x)$ are twice continuously differentiable and the derivative of $CO(t)$ given by (2.5) is Lipschitz continuous with a constant L. If $\{\sigma_k\}$ is bounded away from zero, then the sequence of $\{\bar{\alpha}_k\}$ generated by the PCNC algorithm is bounded away from zero.*

Proof. Since $\bar{\alpha}_k = \min(\alpha_k^I, \alpha_k^{II})$ it sufices to show that the sequences $\{\alpha_k^I\}$ and $\{\alpha_k^{II}\}$ are bounded away from zero. Let us suppress the subscript k. Following El-Bakry, Tapia, Tsuchiya and Zhang [1996] let us define the vectors: $a(\alpha) = [s(\alpha)\; w(\alpha)\; v(\alpha)]^T \in R^{m+2n}$ and $b(\alpha) = [z(\alpha)\; p(\alpha)\; q(\alpha)]^T \in R^{m+2n}$. Hence, the function $\Theta^I(\alpha)$ from (6.2) it is written as:

$$\Theta^I(\alpha) = \frac{\min\{a_i(\alpha)b_i(\alpha)\}}{\frac{a^T(\alpha)b(\alpha)}{2n+m}} - \gamma\tau_1.$$

From the definition of α^I (step 5c) we see that α is the largest number in $[0, \alpha_m]$ such that

$$a_i(\alpha)b_i(\alpha) - \gamma\tau_1 a^T(\alpha)b(\alpha)/(2n+m) \geq 0,$$

for every $i = 1, ..., 2n+m$ and $\alpha \in [0, \alpha^I]$.
Let $\eta_i = \mid \Delta a_i \Delta b_i - \gamma\tau_1 \frac{\Delta a^T \Delta b}{2n+m} \mid$. From proposition 8 it follows that Δt is bounded, then there is a positive constant M such that $\eta_i \leq M$. A straightforward computation shows that, for $\alpha \in [0, \alpha^I]$,

$$\begin{gathered}
a_i(\alpha)b_i(\alpha) - \gamma\tau_1 \frac{a^T(\alpha)b(\alpha)}{2n+m} \\
= \left[a_i b_i - \gamma\tau_1 \frac{a^T b}{2n+m}\right] + \alpha\left[a_i\Delta b_i + b_i \Delta a_i - \gamma\tau_1 \frac{a^T\Delta b + b^T \Delta a}{2n+m}\right] + \alpha^2\left[\Delta a_i \Delta b_i - \gamma\tau_1 \frac{\Delta a^T \Delta b}{2n+m}\right] \\
\geq \alpha(1-\gamma\tau_1)\mu - \alpha^2 \mid \Delta a_i \Delta b_i - \gamma\tau_1 \frac{\Delta a^T \Delta b}{2n+m} \mid \\
= \alpha(1-\gamma\tau_1)\mu - \eta_i \alpha^2 \\
\geq \alpha(1-\gamma\tau_1)\mu - M\alpha^2 \geq 0.
\end{gathered}$$

Hence,

$$\alpha^I \geq (1 - \gamma\tau_1)\mu/M.$$

But, as we know, $\mu = \sigma \frac{a^T b}{2n+m}$ and for σ bounded away from zero it follows that μ is bounded below. Hence, α^I is bounded away from zero.

Now, let us show that $\{\alpha_k^{II}\}$ generated by the step 5 of the algorithm PCNC is bounded away from zero. By the mean-value theorem for vector-valued function [Dennis and Schnabel, 1983, Chap. 4] we have:

$$\begin{gathered}
CO(t + \alpha\Delta t) = CO(t) + \alpha \left[\int_0^1 \nabla CO(t + \xi\alpha\Delta t) d\xi \right] \Delta t \\
= CO(t) + \alpha \nabla CO(t) \Delta t + \alpha \left[\int_0^1 \left(\nabla CO(t + \xi\alpha\Delta t) - \nabla CO(t) \right) d\xi \right] \Delta t \\
= (1 - \alpha) CO(t) + \alpha \left[\int_0^1 \left(\nabla CO(t + \xi\alpha\Delta t) - \nabla CO(t) \right) d\xi \right] \Delta t,
\end{gathered}$$

where the last equality is from (3.1). Now, having in view that the derivative of $CO(t)$ is Lipschitz continous with the constant L, we obtain:

$$\| CO(t + \alpha\Delta t) \| \leq \| CO(t) \| | 1 - \alpha | + L\alpha^2 \| \Delta t \|^2 .$$

Using this inequality we have:

$$\begin{gathered}
\Theta^{II}(\alpha) = a^T(\alpha) b(\alpha) - \gamma\tau_2 \| CO(t + \alpha\Delta t) \| \\
= a^T b + \alpha \left[a^T \Delta b + b^T \Delta a \right] + \alpha^2 \Delta a^T \Delta b - \gamma\tau_2 \| CO(t + \alpha\Delta t) \| \\
\geq | 1 - \alpha | a^T b + \alpha\sigma a^T b + \alpha^2 \Delta a^T \Delta b - \gamma\tau_2 \left(\| CO(t) \| | 1 - \alpha | + L\alpha^2 \| \Delta t \|^2 \right) \\
= | 1 - \alpha | \left(a^T b - \gamma\tau_2 \| CO(t) \| \right) + \alpha\sigma a^T b + \alpha^2 \left(\Delta a^T \Delta b - \gamma\tau_2 L \| \Delta t \|^2 \right) \\
\geq \alpha \left[\sigma a^T b - \alpha | \Delta a^T \Delta b - \gamma\tau_2 L \| \Delta t \|^2 | \right].
\end{gathered}$$

From proposition 8 there is a constant $N > 0$ such that

$$| \Delta a^T \Delta b - \gamma\tau_2 L \| \Delta t \|^2 | \leq N.$$

Hence,

$$\Theta^{II}(\alpha) \geq \alpha \left[\sigma a^T b - \alpha N \right].$$

From condition (6.15) it follows that

$$\alpha^{II} \geq \sigma a^T b / N.$$

Since $\{\sigma_k\}$ is bounded away from zero, then the sequence $\{\alpha_k^{II}\}$ is bounded away from zero. $\square$

Theorem 4. *Suppose that the functions of the problem $f(x)$, $g(x)$ and $h(x)$ are twice continuously differentiable and the derivative of $CO(t)$ given by (2.5) is Lipschitz continuous. Let $\{t_k\}$ be generated by the algorithm PCNC, where $\{\sigma_k\} \subset (0,1)$ is bounded away from zero and one. Then the sequence $\{F(t_k)\}$ converges to zero, and for any limit point $t^* = [x^*, y^*, z^*, p^*, q^*, s^*, w^*, v^*]^T$, x^* is a KKT point of the problem (1).*

Proof. Firstly, note that the sequence $\{\| F(t_k) \|\}$ is monotone decreasing, hence it is convergent. By contradiction, suppose that $\{\| F(t_k) \|\}$ is not convergent to zero. But, from proposition 5 we have

$$\Phi_k(\alpha_k)/\Phi_k(0) \leq 1 - 2\alpha_k \beta(1-\sigma_k)$$

Then, from proposition 13 it follows that the sequence $\{\Phi_k\}$ converges linearly to zero. This gives a contradiction.

On the other hand, from proposition 2 we have:

$$\nabla\Phi(t_k)\Delta t_k = -2F(t_k)^T F(t_k) + 2\mu_k F(t_k)\,\hat{e}\,.$$

Since the sequence $\{\alpha_k\}$ is bounded away from zero, then the backtracking line search used in step 5 of the algorithm PCNC produces:

$$\frac{\nabla\Phi(t_k)\Delta t_k}{\|\Delta t_k\|} = \frac{-2[F(t_k)^T F(t_k) - \mu_k F(t_k)\hat{e}]}{\|\Delta t_k\|} \to 0.$$

From proposition 8, Δt_k is bounded away from zero, then

$$\Phi(t_k) - \mu_k[s_k^T z_k + w_k^T p_k + v_k^T q_k] \to 0.$$

However,

$$\Phi(t_k) - \mu_k[s_k^T z_k + w_k^T p_k + v_k^T q_k] \geq (1-\sigma_k)\Phi(t_k).$$

Therefore, it must hold that $\Phi(t_k) \to 0$, because the sequence $\{\sigma_k\}$ is bounded away from one. This again leads to a contradiction. Hence, the sequence $\{\| F(t_k) \|\}$ must converge to zero. Since the KKT conditions for the problem (1) are satisfied by t^*, it follows that x^* is a KKT point of the problem (1). $\square$

8. Numerical Examples

In this section we shall present some numerical results obtained with a crude experimental code which is still in progress of development. Having in view that the conditions (6.17d) and (6.17f) are difficult to be implemented, for step length determination, we shall consider a variant of the algorithm in which only the conditions (6.17a, b, c and e) are implemented. Firstly we present details of the running of the algorithm for a number of three problems. After that, some characteristics of the optimization process with PCNC for some problems from the Hock and Schittkowski's and Schittkowski's sets of test problems are presented.

Example 1. Let us consider the problem: [Hock and Schittkowski, 1981, problem 71, pp.92]

$$\min x_1 x_4 (x_1 + x_2 + x_3) + x_3$$

subject to:

$$\begin{gathered} x_1 x_2 x_3 x_4 - 25 \geq 0, \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 - 40 = 0, \\ 1 \leq x_i \leq 5,\ i = 1, 2, 3, 4. \end{gathered}$$

The evolution of some elements of the algorithm PCNC is given in tables 1 and 2.

Table 1. The evolution of the functions ($\varepsilon = 10^{-8}$):

$$\begin{aligned} f(x_k) &= \textit{value of the objective} \\ \Phi(t_k) &= \textit{value of the merit function} \\ \| CO(t_k) \|_2 &= \textit{norm of the optimality conditions} \\ \| CT(.) \|_2 &= \textit{norm of the transversality conditions} \end{aligned}$$

k	$f(x_k)$	$\Phi(t_k)$	$\| CO(t_k) \|_2$	$\| CT(.) \|_2$
0	.160000E+02	.534076E+03	.465076E+03	.690000E+02
1	.169494E+02	.112666E+02	.897539E+01	.229126E+01
2	.171584E+02	.253076E+00	.897539E+01	.135262E+00
3	.169951E+02	.202962E-01	.141485E+01	.614777E-02
4	.170168E+02	.455528E-03	.188750E-03	.266778E-03
5	.170158E+02	.110764E-04	.526546E-06	.105499E-04
6	.170144E+02	.424528E-06	.561720E-09	.423877E-06
7	.170141E+02	.162452E-07	.651720E-09	.162445E-07
8	.170140E+02	.243738E-10	.199940E-14	.243718E-10

Table 2. The evolution of the parameters ($\varepsilon = 10^{-8}$):

$$\begin{aligned} \gamma_k &= \textit{value of parameter } \gamma \textit{ in function } \Theta^I(\alpha\ ) \\ \sigma_k &= \textit{value of parameter } \sigma \textit{ for barrier parameter } \mu \\ \alpha_{\max} &= \textit{maximum value of the steplength} \\ \alpha_k &= \textit{value of steplength} \\ \mu_k &= \textit{value of the barrier parameter} \end{aligned}$$

k	γ_k	σ_k	$\alpha_{\max}$	α_k	μ_k
0	0.750000	0.1	1.090858	1.039720	.233333E+00
1	0.625000	0.2	0.992283	0.926926	.906640E-01
2	0.562500	0.2	1.170237	1.0	.226439E-01
3	0.531250	0.2	1.210422	1.0	.488023E-02
4	0.515625	0.2	1.165723	1.0	.105805E-02
5	0.507812	0.2	1.220644	1.0	.216128E-03
6	0.503906	0.19530	1.238018	1.0	.423836E-04
7	0.501953	0.03823	1.039720	1.0	.162444E-05

Some remarks are in order:

1. The parameter γ_k is updated as: $\gamma_{k+1} = 0.5 + (\gamma_k - 0.5)/2$. Clearly, we can imagine some other updating formulas for $\gamma_k \in [0.5, \gamma_{k-1}]$.

2. The parameter σ_k is updated as:

$$\sigma_k = \begin{cases} \eta_1, & if \sigma_k \leq \eta_2(s^T z + w^T p + v^T q), \\ \eta_2(s^T z + w^T p + v^T q), & if \sigma_k > \eta_2(s^T z + w^T p + v^T q) \end{cases}$$

where $\eta_1 = 0.1$ and $\eta_2 = 100$.

3. Even that the (6.17d) condition: $\Theta^{II}(\alpha_k) \geq 0$ has not been implemented we see that in the last part of the optimization process: $\| CO(t_k) \|_2 < \| CT(.) \|_2$, this ensuring the convergence of the algorithm.

4. Note that the steplength $\alpha_k \to 1$, exactly as in the "pure" Newton method.

5. It is interesting to see the evolution of the barrier parameter μ_k given by (5.4) or by (6.8). Table 3 illustrates this behaviour.

Table 3. The evolution of the barrier papameter μ_k ($\varepsilon = 10^{-8}$)

k	μ_k given by (6.8)	μ_k given by (5.4)
0	.233333E+00	.254322E+02
1	.906640E-01	.276151E+01
2	.226439E-01	.248363E+00
3	.488023E-02	.924194E-01
4	.105805E-02	.956741E-02
5	.216128E-03	.113888E-02
6	.423836E-04	.217363E-03
7	.162444E-05	.424866E-04

6. The solution of this problem is $x = [1\ 5\ 5\ 1]^T$. The value of the objective function is 17.0140173. The value of the equality constraints is .7638083E-08, and of the inequality constraints is .2935759E-05.

7. The SPENBAR package [Andrei, 1996a-d] gives a solution for this problem involving 8 major iterations, 143 minor iterations (truncated Newton iterations) and 591 evaluations of the functions of the problem.

Example 2. Let us consider the problem [Hock and Schittkowski, 1981, problem 43, pp.66, (Rosen -Suzuki)]:

$$\min x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4$$

subject to:

$$\begin{gathered} 8 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 \geq 0, \\ 10 - x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_4 \geq 0, \\ 5 - 2x_1^2 - x_2^2 - x_3^2 - 2x_1 + x_2 + x_4 \geq 0, \\ -2 \leq x_i \leq 5,\ i = 1, 2, 3, 4. \end{gathered}$$

The algorithm PCNC gives the following solution in table 4:

Table 4. Solution of the problem ($\varepsilon = 10^{-8}$)

#	l	x^*	u
1	-2.0	.992394E-12	5.0
2	-2.0	1.0	5.0
3	-2.0	2.0	5.0
4	-2.0	-1.0	5.0

The evolution of some elements of the algorithm PCNC is given in tables 5 and 6.

Table 5. The evolution of the functions ($\varepsilon = 10^{-8}$):

$$\begin{aligned} f(x_k) &= \textit{value of the objective} \\ \Phi(t_k) &= \textit{value of the merit function} \\ \| CO(t_k) \|_2 &= \textit{norm of the optimality conditions} \\ \| CT(.) \|_2 &= \textit{norm of the transversality conditions} \end{aligned}$$

k	$f(x_k)$	$\Phi(t_k)$	$\| CO(t_k) \|_2$	$\| CT(.) \|_2$
0	.000000E+00	.737000E+03	.536000E+03	.201000E+03
1	-.977641E+01	.549977E+03	.401579E+03	.148398E+03
2	-.371578E+02	.100816E+03	.528640E+02	.479521E+02
3	-.445550E+02	.783561E+01	.259689E+01	.523872E+01
4	-.437880E+02	.341158E+00	.413957E-01	.299762E+00
5	-.439719E+02	.322406E-01	.100845E-02	.312322E-01
6	-.439805E+02	.136610E-02	.150560E-03	.121554E-02
7	-.439962E+02	.480056E-04	.819310E-07	.479236E-04
8	-.439991E+02	.191398E-05	.441733E-10	.191394E-05
9	-.439998E+02	.768899E-07	.259717E-13	.768898E-07
10	-.439999E+02	.656910E-09	.612515E-16	.656910E-09
11	-.440000E+02	.531802E-13	.681308E-20	.531802E-13

Table 6. The evolution of the parameters ($\varepsilon = 10^{-8}$):

$$\begin{aligned} \gamma_k &= \textit{value of parameter } \gamma \textit{ in function } \Theta^I(\alpha\ ) \\ \sigma_k &= \textit{value of parameter } \sigma \textit{ for barrier parameter } \mu \\ \alpha_{\max} &= \textit{maximum value of the steplength} \\ \alpha_k &= \textit{value of steplength} \\ \mu_k &= \textit{value of the barrier parameter} \end{aligned}$$

k	γ_k	σ_k	$\alpha_{\max}$	α_k	μ_k
0	.750000	.3	0.173623	0.173620	.106363E+01
1	.625000	.2	0.610374	0.610371	.615770E+00
2	.562500	.2	0.776175	0.776172	.350629E+00
3	.531250	.2	1.099367	1.000000	.116627E+00
4	.515625	.2	0.888802	0.888801	.284905E-01
5	.507812	.2	1.170841	1.0	.888200E-02
6	.503906	.2	1.164319	1.0	.203728E-02
7	.501953	.2	1.193870	1.0	.415577E-03
8	.500976	.2	1.243771	1.0	.834214E-04
9	.500488	.91966E-01	1.101126	1.0	.768897E-05
10	.500244	.85005E-02	1.008572	1.0	.656909E-07
11	.500122	.76484E-04	1.000076	1.0	.531801E-11

The SPENBAR package gives the same solution involving 5 major iterations, 85 minor iterations and 347 functions evaluations.

Example 3. Let us consider the problem [Hock and Schittkowski, 1981, problem 113, pp.122, (Wong No. 2)]:

$$\min x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 +$$
$$(x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 +$$
$$(x_{10} - 7)^2 + 45$$

subject to:

$$\begin{gathered}
105 - 4x_1 - 5x_2 + 3x_7 - 9x_8 \geq 0,\\
-10x_1 + 8x_2 + 17x_7 - 2x_8 \geq 0,\\
8x_1 - 2x_2 - 5x_9 + 2x_{10} + 12 \geq 0,\\
-3(x_1 - 2)^2 - 4(x_2 - 3)^2 - 2x_3^2 + 7x_4 + 120 \geq 0,\\
-5x_1^2 - 8x_2 - (x_3 - 6)^2 + 2x_4 + 40 \geq 0,\\
-3x_5^2 - 0.5(x_1 - 8)^2 - 2(x_2 - 4)^2 + x_6 + 30 \geq 0,\\
-x_1^2 - 2(x_2 - 2)^2 + 2x_1x_2 - 14x_5 + 6x_6 \geq 0,\\
3x_1 - 6x_2 - 12(x_9 - 8)^2 + 7x_{10} \geq 0,\\
0 \leq x_i \leq 10,\ i = 1, ..., 10.
\end{gathered}$$

The algorithm PCNC gives the following solution in table 7:

Table 7. Solution of the problem ($\varepsilon = 10^{-8}$)

#	l	x^0	x^*	u
1	0.0	2.0	.2171996E+01	10.0
2	0.0	3.0	.2363683E+01	10.0
3	0.0	5.0	.8773926E+01	10.0
4	0.0	5.0	.5095984E+01	10.0
5	0.0	1.0	.9906548E+01	10.0
6	0.0	2.0	.1430574E+01	10.0
7	0.0	7.0	.1321644E+01	10.0
8	0.0	3.0	.9828726E+01	10.0
9	0.0	6.0	.8280092E+01	10.0
10	0.0	10.0	.8375927E+01	10.0

Table 8. The value of the constraints ($\varepsilon = 10^{-8}$)

i	$h_i(x^*)$
1	.3936866E-09
2	.1424133E-08
3	.4911662E-09
4	.3292851E-07
5	.2165631E-08
6	.6148503E+01
7	.2354256E-08
8	.5002396E+02

The value of the objective function is: 24.30621.

The evolution of some elements of the algorithm PCNC is given in tables 9 and 10.

Table 9. The evolution of the functions ($\varepsilon = 10^{-8}$):

$$
\begin{aligned}
f(x_k) &= \textit{value of the objective} \\
\Phi(t_k) &= \textit{value of the merit function} \\
\| CO(t_k) \|_2 &= \textit{norm of the optimality conditions} \\
\| CT(.) \|_2 &= \textit{norm of the transversality conditions}
\end{aligned}
$$

k	$f(x_k)$	$\Phi(t_k)$	$\| CO(t_k) \|_2$	$\| CT(.) \|_2$
0	.753000E+03	.451040E+05	.442410E+05	.863000E+03
1	.576916E+03	.318551E+05	.312051E+05	.649950E+03
2	.221882E+03	.760135E+04	.740377E+04	.197577E+03
3	.790325E+02	.915963E+03	.878209E+03	.377534E+02
4	.447106E+02	.342317E+03	.331333E+03	.109840E+02
5	.265738E+02	.270729E+02	.261570E+02	.915983E+00
6	.245296E+02	.592638E-01	.162438E-01	.430200E-01
7	.243544E+02	.186719E-02	.340773E-04	.183312E-02
8	.243160E+02	.742385E-04	.109610E-06	.741289E-04
9	.243081E+02	.298381E-05	.102978E-09	.298371E-05
10	.243066E+02	.119888E-06	.106694E-12	.119888E-06
11	.243062E+02	.404292E-08	.171373E-15	.404292E-08
12	.243062E+02	.470981E-11	.376155E-18	.470981E-11

Table 10. The evolution of the parameters ($\varepsilon = 10^{-8}$):

γ_k = *value of parameter* γ *in function* $\Theta^I(\alpha\ )$

σ_k = *value of parameter* σ *for barrier parameter* μ

$\alpha_{\max}$ = *maximum value of the steplength*

α_k = *value of steplength*

μ_k = *value of the barrier parameter*

k	γ_k	σ_k	$\alpha_{\max}$	α_k	μ_k
0	.750000	.3	0.158304	0.158304	.140357E+01
1	.625000	.2	0.502918	0.502918	.809315E+00
2	.562500	.2	0.657969	0.657969	.435822E+00
3	.531250	.2	0.535589	0.535589	.200213E+00
4	.515625	.2	0.890842	0.890842	.113156E+00
5	.503906	.2	1.127216	1.0	.341684E-01
6	.503906	.2	1.209376	1.0	.764177E-02
7	.501953	.2	1.170946	1.0	.160451E-02
8	.500976	.2	1.222116	1.0	.325237E-03
9	.500488	.2	1.244544	1.0	.652865E-04
10	.500244	.18321	1.223653	1.0	.119888E-04
11	.500122	.33645	1.034818	1.0	.404292E-06
12	.500061	.11483	1.001150	1.0	.470981E-09

In this case SPENBAR package requires 4 major iterations, 149 minor (truncated Newton) iterations and 970 evaluations of functions.

In the following we shall consider the running of the PCNC package on a number of problems from the Hock - Schittkowski [1981] set of problems, as well as from Schittkowski's [1987] collection of problems. Table 11 shows the characteristics of the optimization process with PCNC refering to different initial points, number of iterations, number of functions evaluations, the value of the norm of the optimality conditions, the value of the barrier parameter and of the steplength, etc.

Table 11. Characteristics of the optimization process with PCNC ($\varepsilon = 10^{-8}$):

$$\begin{aligned}
IP &= \textit{Initial Point (S = Standard Point)} \\
nit &= \textit{\# of iterations} \\
nf &= \textit{\# of functions evaluations} \\
\| CO(t_k) \|_2 &= \textit{Norm of optimality conditions} \\
\| CT(.) \|_2 &= \textit{Norm of centrality conditions} \\
f &= \textit{Value of the objective function} \\
\xi &= \textit{Distance from centrality} \\
\mu &= \textit{Value of the barrier parameter} \\
\alpha &= \textit{Value of the steplength} \\
\alpha_{\max} &= \textit{Maximum value of the steplength}
\end{aligned}$$

Name	*IP*	*nit*	*nf*	$\| CO(t_k) \|_2$	$\| CT(.) \|_2$	f	ξ	μ	α	$\alpha_{\max}$
HS32	S	6	8	.281190E-09	.890623E-5	1.0	.673457	.747336E-03	0.999500	1.287417
	I1	6	8	.827097E-08	.906026E-04	1.0	.631756	.281243E-02	0.999500	1.219009
	I2	35	37	.238509E-09	.706660E-05	1.0	..622537	.626097E-02	0.999500	1.266965
	I3	36	38	.542306E-08	.512380E-04	1.0	.704269	.184727E-02	0.999500	1.229708
HS33	S	61	63	.723080E-08	.346289E-11	1.798992	.00022	.374461E-08	0.717951	0.718310
	I1	57	59	.238560E-08	.186609E-12	1.798992	.0.00047	.141810E-08	0.893644	0.894091
HS37	I1	14	16	.206426E-27	.228163E-20	-3456.0	0.999999	.585360E-12	0.999397	1.000
HS42	S	10	12	.476987E-26	.106282E-16	13.85786	0.99995	.688951E-09	0.9995	1.00074
	I1	9	11	.263809E-24	.572985E-15	13.85786	0.999949	.698862E-08	0.9995	1.00237
	I2	10	12	.232637E-28	.325052E-19	13.85786	0.99995	.943834E-11	0.9995	1.00008
	I3	9	11	.137112E-24	.297692E-15	13.85786	0.999949	.486897E-08	0.9995	1.00197
HS43	S	13	15	.277706E-09	.302020E-09	-44.0	.282086	.668179E-08	0.135259	1.00829
	I1	12	14	.134241E-09	.139405E-09	-44.0	0.166757	.879352E-08	0.630348	1.014314
	I2	12	14	.246198E-09	.256468E-09	-44.0	0.177464	.531238E-08	0.212828	1.012284
	I3	13	15	.121085E-09	.127335E-09	-44.0	0.203728	.212161E-08	0.086971	1.006461
HS44	I1	12	14	.789613E-29	.906522E-19	-15.0	.999473	.202883E-10	0.999500	1.000169
	I2	12	14	.342044E-28	.539278E-18	-15.0	.999473	.781634E-10	0.999500	1.000331
HS47	I1	55	57	.867998E-08	.883014E-41	0.0	.04927	.546914E-17	0.938531	0.999661
	I2	54	56	.307564E-04	.267626E-18	.133442E-03	0.05	.221299E-16	.312620E-03	0.991663
HS48	S	4	6	.981479E-13	.380432E-01	0.0	.999169	.615545E-04	0.999500	1.223207
	I1	4	6	.935982E-13	.376255E-01	0.0	.999211	.612159E-04	0.999500	1.222694
	I2	4	6	.328648E-10	.460264E-01	0.0	.990773	.213457E-03	0.999500	1.140494
HS49	S	15	17	.914903E-08	.240757E-11	0.0	.0495064	.180460E-10	0.093525	0.997787
	I1	26	28	.277362E-08	.171366E-19	0.0	.150	.222354E-16	0.765397	0.997641

Table 11. Characteristics of the optimization process with PCNC ($\varepsilon = 10^{-8}$) - continued

Name	*IP*	*nit*	*nf*	$\|\| CO(t_k) \|\|_2$	$\|\| CT(.) \|\|_2$	f	ξ	μ	α	$\alpha_{\max}$
HS50	S	10	12	.207630E-09	.231393E-06	0.0	.999374	.151811E-03	0.999500	1.245079
	I1	9	11	.180172E-11	.123154E-05	0.0	.999917	.350233E-03	0.999500	1.247689
	I2	14	16	.275918E-13	.567711E-20	0.0	.883577	.190954E-11	0.999500	1.000040
	I3	6	8	.114298E-09	.968789E+00	.487294E-07	.999076	.310624E+00	0.999500	1.210478
	I4	6	8	.644158E-08	.250051E+00	.205908E-06	.997404	.498941E+00	0.999500	1.183567
HS51	S	4	6	.385563E-12	.115810E-01	.331885E-05	.948848	.334951E-01	0.999500	1.149867
	I1	4	6	.551270E-11	.283667E-01	.113598E-04	.934210	.522127E-01	0.999500	1.162979
	I2	4	6	.175057E-11	.196812E-01	.348339E-06	.959153	.441287E-01	0.999500	1.150994
HS52	S	2	4	.347500E-09	.161794E+03	5.326641	.999659	.401433E+01	0.999500	1.219962
	I1	3	5	.195840E-08	.423945E+02	5.326636	.999791	.205487E+01	0.999500	1.229861
	I2	3	5	.223451E-08	.426037E+02	5.326659	.999790	.205994E+01	0.999500	1.229707
	I3	8	10	.116096E-13	.311722E+01	5.326648	.999996	.557206E+00	0.999500	1.249595
HS53	S	3	5	.155242E-10	.395810E+00	4.093025	.996253	.198503E+00	0.999500	1.160760
	I1	4	6	.135780E-08	.223385E+01	4.093031	.996831	.471426E+00	0.999500	1.164569
	I2	4	6	.247691E-08	.257194E+01	4.093054	.992703	.505691E+00	0.999500	1.169289
	I3	3	5	.134293E-11	.729796E-01	4.093024	.998795	.852519E-01	0.9995000	1.232501
HS61	I1	9	11	.255905E-10	.485008E-08	-143.64614	.999699	.283372E-04	0.999500	1.144733
HS63	S	6	8	.307336E-10	.155108E-04	961.7152	.992672	.160669E-02	0.999500	1.231509
	I1	6	8	.707607E-08	.124090E-04	961.7151	.934392	.145152E-02	0.999500	1.174307
	I2	9	11	.511842E-11	.744801E-05	961.7152	.999600	.111198E-02	0.999500	1.239978
HS65	S	7	9	.276898E-09	.102787E-04	.9547013	.966343	.120727E-02	0.999500	1.215245
	I1	7	9	.673553E-11	.123496E-05	.9539442	.988519	.418477E-03	0.999500	1.208355
	I2	5	7	.648425E-08	.283699E-04	.9554332	.946089	.198347E-02	0.999500	1.187995
HS71	S	6	8	.651720E-09	.423877E-06	17.01444	.984698	.216128E-03	0.999500	1.220644
	I1	7	9	.435015E-08	.884619E-06	17.01462	.975000	.311775E-03	0.999500	1.214727
HS73	S	8	10	.176433E-10	.173662E-05	29.89562	.964530	.416176E-03	0.999500	1.097279
	I1	10	12	.513410E-09	.120897E-08	29.89441	.995412	.109476E-04	0.999500	1.115710
	I2	9	11	.339001E-08	.442336E-06	29.89500	.970094	.209943E-03	0.999500	1.143323
	I3	10	12	.712451E-09	.290912E-06	29.89489	.994730	.170205E-03	0.999500	1.136257
HS76	S	10	12	.140067E-29	.343496E-16	-4.681818	.998205	.123711E-08	0.999500	1.001168
	I1	10	12	.578975E-29	.752265E-15	-4.681818	.998201	.700899E-08	0.999500	1.002785
	I2	11	13	.108198E-29	.104317E-20	-4.681818	.998200	.384940E-12	0.999500	1.000021
	I3	12	14	.188957E-29	.814056E-17	-4.681818	.998207	.517577E-09	0.999500	1.000755
HS113	S	13	15	.366188E-25	.487388E-18	24.30621	.999871	.592351E-10	0.999500	1.000407
	I1	17	19	.167600E-22	.204212E-15	24.30621	.999870	.225305E-08	0.999500	1.002518
	I2	12	14	.176304E-23	.224720E-16	24.30621	.999870	.654417E-09	0.999500	1.001356
	I3	18	20	.248136E-25	.269701E-18	24.30621	.999871	.390857	0.999500	1.000331
S217	S	13	15	.749654E-24	.615541E-19	-0.8	.997026	.174759E-10	0.999500	1.000094
	I1	9	11	.112356E-23	.923683E-19	-0.8	.997027	.247263E-10	0.999500	1.000111
S218	S	14	16	.268757E-10	.314477E-07	.774652E-07	.0220820	.105397E-08	0.225725	1.031481
S219	I1	14	16	.388070E-10	.163815E-09	-0.999999	.100000	.570597E-08	0.089401	1.021684

Table 12 shows the comparative results of the PCNC and SPENBAR results of the set of problems considered in this experiment. In PCNC code the Newton method is used as the basic iterative procedure. In SPENBAR we used the truncated Newton method [Nash, 1985], [Nash, Polyak and Sofer, 1994]. In the modified penalty-barrier code SPENBAR, the number of major iterations *(o-it)* is the number of times μ (the barrier parameter) was decreased and the Lagrange multipliers adjusted. The number of minor iterations *(in-it)* is the total number of truncated Newton steps. For the PCNC, the number of iterations is the total number of Newton steps. In both codes , *nf* is the total number the functions of the problem have been evaluated.

Table 12: Comparison between SPENBAR and PCNC

Problem	n	m	me	SPENBAR			PCNC	
				o-it	in-it	nf	it	nf
HS32	3	1	1	10	136	348	6	8
HS33	3	2	0	10	66	187	61	63
HS37	3	2	0	4	26	76	14	16
HS42	4	0	2	5	23	84	10	12
HS43	4	3	0	5	85	347	13	15
HS44	4	6	0	9	107	498	12	14
HS47	5	0	3	8	28	278	55	57
HS48	5	0	2	4	50	179	4	6
HS49	5	0	2	3	126	459	15	17
HS50	5	0	3	4	106	388	10	12
HS51	5	0	3	5	91	324	4	6
HS52	5	0	3	6	81	280	2	4
HS53	5	0	3	6	74	284	3	5
HS61	3	0	2	4	37	96	9	11
HS63	3	0	2	4	51	153	6	8
HS65	3	1	0	5	203	659	7	9
HS71	4	1	1	8	143	591	6	8
HS73	4	2	1	7	285	1128	8	10
HS76	4	3	0	9	96	364	10	12
HS113	10	8	0	4	149	970	13	15
S217	2	1	1	5	33	83	13	15
S218	2	1	0	9	50	234	14	16
S219	4	0	2	6	106	401	14	16
TOTAL				140	2152	8411	309	355

Computational experience with this approach is very limited. Using the same primal-dual interior-point approach, El-Bakry, Tapia, Tsuchiya and Zhang [1996] report some preliminary computational results on alimited number of test problems of small dimensions from the Hock and Schittkowski collection. Lasdon, Plummer and Yu [1995] report results on a somewhat larger test set, of a variant of algorithm using a trust region approach. Vanderbei and Shanno [1997] present results on the Hock - Schittkowski problems as well as on a number of 8 large-scale nonlinear problems in comparison with MINOS and LANCELOT.

Gay, Overton and Wright [1997] also report results on the Hock - Schittkowski problems and three problems from the CUTE [1995] collection. We have considered here some numerical experience with a variant of the algorithm which is a slight simplification, i.e. we did not enforced conditions (6.17d and f) for steplength determination in order to avoid the complications regarding the nonlinear function $\Theta^{II}(\alpha)$. A robust implementation must consider these conditions in the line search procedure from set 5 of the algorithm. The algorithm contains a number of parameters, and its performance is very dependent upon their choice. To choose proper values of these parameters require very intensive numerical tests.

9. Conclusion

In this paper we have presented an interior point algorithm for solving general nonlinear programming problems. It is based on the perturbed Karush-Kuhn-Tucker conditions associated to the logarithmic barrier function formulation of the problem.

The algorithm has two main components: determination of the descent direction and the computation of the steplength. The search direction is computed as solution of a reduced perturbed Newton system. Vanderbei and Shanno [1997] show that for nonconvex nonlinear problems this system must be modified by adding a diagonal matrix. We did not considered this perturbation in our implementation. The critical feature of the algorithm is the choices of the steplengths α_k at each iteration in order to find the local minimizer of (1). For this, a number of criteria must be implemented (see (6.17)), the main of which is that refering to the speed of reduction to zero of the pure optimality conditions versus the transversality conditions. The convergence is assured when both these conditions converge to zero, but in such a proportion that the transverality conditions are greater that the pure optimality conditions. In our experimental code we did not implemented this condition.

Mainly, the algorithm does not mimic the primal-dual interior-point methods for linear programming. Rather, some special elements have been introduced as: formulation of the optimality conditions, definition of the perturbed reduced Newton system, definition of a merit function, choice of the barrier parameter. Some other points remain to be clarified: detection and treatment of the indefinitness in the Newton system, selection of proper values of the parameters β, γ_k, σ_k and ρ, treatment of an infeasible starting point, practical implementation of the conditions (6.17) for steplength determination. All of these issues deserve further research and experimentation.

The results of this preliminary computational experimentation indicate that the primal-dual approach is more efficient than the penalty-barrier approach. However, the primal-dual algorithm requires second partial derivatives of the functions of the problem. Thus, for that problems for which the second derivative information is difficult to obtain, the penalty-barrier method should be preferable. An advantage of the primal-dual approach is that the Lagrange multipliers are computed directly by the Newton's method rather than using the first order estimates. This is very appealing improving both accuracy and the rate of convergence of the algorithm.

In summary, the reconsideration of the barrier methods for nonlinear programming is one of the very active area of research. Much remains to be done both from theoretical view point and computational, in order to build up a viable algorithm and an efficient implementation able to resist at the numerical comparisons with the state-of-the-art packages like: MINOS, LANCELOT, SNOPT, NLPQL, SPENBAR, CONOPT, etc.

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APPENDIX

The starting points indicated in table 11 are the following:

$Problem$	$Initial - Points$
HS32	I1=[1,7,2];I2=[9,9,2];I3=[9,9,9]
HS33	I1=[0,1,1]
HS37	I1=[30,30,30]
HS42	I1=[2,2,2,2];I2=[3,3,3,3];I3=[3,3,1,1]
HS43	I1=[1,0,0,0];I2=[1,1,0,0,];I3=[1,1,1,1]
HS44	I1=[1,2.5,0.5,3.5];I2=[1,4.5,0.5,3.5]
HS47	I1=[0.5,0.5,0.5,0.5,0.5];I2=[1,0.5,1,0.5,1]
HS48	I1=[3,5,-3,2,-1];I2=[1,7,2,-3,5]
HS49	I1=[9,8,1,-4,0.5]
HS50	I1=[10,-30,11,15,-15];I2=[30,-30,35,15,-35]; I3=[35,35,35,35,35];I4=[-35,-35,-35,-35,-35]
HS51	I1=[3,2.5,2,-1,2.5];I2=[-2,2.5,-2,-1,3]
HS52	I1=[20,20,20,20,20];I2=[-20,-20,-20,-20,-20] I3=[200,-200,200,-200,200]
HS53	I1=[10,10,10,10,10];I2=[-10,-10,-10,-10,-10] I3=[0,0,0,0,0]
HS61	I1=[3,-5,1]
HS63	I1=[3,3,3];I2=[1,1,1]
HS65	I1=[-5,4,0];I2=[5,4,0]
HS71	I1=[2,5,5,2]
HS73	I1=[2,1,1,0.5];I2=[2,2,2,0.5];I3=[2,2,2,2]
HS76	I1=[1,1,1,1];I2=[2,2,2,2];I3=[5,5,5,5]
HS113	I1=[10,3,1,1,1,2,4,3,6,10]; I2=[2,2,9,6,1,2,2,10,10,10]; I3=[20,20,20,20,20,20,20,20,20,20]
S217	I1=[1,1]
S219	I1=[1,1,1,1]