# An Inventory Model for a Single Item with Different Types of Demand Under Different System of Management 

Jayanta Kumar Dey ${ }^{1}$, Manoranjan Maiti. ${ }^{2}$<br>${ }^{1}$ Department of Mathematics, Mahishadal Raj College, Mahishadal, East-Midnapore-721628, India.<br>${ }^{2}$ Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore, 721102 , India


#### Abstract

An inventory problem of a single item with multi-distribution system is considered under different management systems. There are a main warehouse with a sales depot in a central place of a country and one secondary warehouse with one sale cum showroom in each of the n-states. The items from main warehouse are transferred to secondary warehouses in bulk release pattern whereas units are transferred from secondary warehouses to showrooms for sale in continuous or bulk release pattern. The demands at different places are different, may be constant or both stock and price dependent. Here time period for the whole system is assumed to be same and accordingly, there may be different scenarios depending upon the time of exhaustion of the materials at different places. If the item is exhausted early in a place, shortages are allowed at that counter. On the other hand, if there is a surplus at a place, the units are sold at reduced rate and there is an unlimited market for the lower-priced units. Moreover, the whole system is assumed to be both integrated and non-integrated ones. In non-integrated systems, secondary warehouses with their related show rooms are considered to be under separate management houses. Following these assumptions, the inventory model is formulated and solved using a gradient-based non-linear optimization method. The models for different scenarios are illustrated numerically.


## AMS Mathematics Subject Classification: 90B05

## Keywords

Multi-warehouse, stock-dependent demand, constant demand, inventory problem, bulk release pattern, continuous release pattern.

## 1. Introduction

In reality there are many situations in business sector where the demand rate is not constant but varies. It may depend on time, initial or on hand inventory levels, selling price, advertisement expenditure, the frequency of advertisement etc. There are certain types of items (like consumer goods, fashionable items etc.) for which, according to market research, customers are motivated by the display of the items in the showrooms i.e., the demand rate is dependent on the displayed inventory level. For these items, the consumption goes up if the inventory level is high and vice versa. Such type of demand was considered by Gupta and Vrat [1], Mondal and Phaujdar [2,3], Urban [4,5], Bhuina and Maiti[6] etc.

However, in the present competitive market, the selling price is one of the decisive factors in selecting an item for use. It is a common practice that the higher selling price of an item negates the demand of that item whereas lower prices have the reverse effect. Hence, it can be realistically assumed
that the demand of an item is a function of selling price, time, current stock level etc jointly or seperately. In this area, one may refer to the works of Urban [4], Goyal and Gunasekaran[7], Abad[8] among others.

In the field of inventory management, an important problem associated with the inventory maintenance is to decide where to stock the goods. In the busy market place like super market, corporation market, municipality market etc. the storage space of a showroom is limited. When an attractive price discount for bulk purchase is available or the cost of procuring goods is higher than the inventory related other costs or there are some problems in frequent procurements or the demand of items is very high or the bulk transport facility is available etc., management then decides to purchase a huge quantity of items at a time. All these units cannot be stored in the existing storage (showroom, i.e., Primary Warehouse-PW) due to limited capacity. Then for storing the excess units, one (sometimes more than one) additional warehouse (called Secondary Warehouse, SW) is hired on a rental basis. These Sw's may be located near the PW or a little away from it. The products are first stored in PW and then the excess amounts are stored in SW. The actual service to the customer is offered at PW only. The units of SW's are transferred to PW in a continuous/bulk release pattern to meet up the demand at PW until the stock in SW's are emptied and lastly the units at PW are released. There are several related papers presented in this area such as Hartley [9], Sarma[10], Goswami and Chaudhuri[11 ], Pakkala Achary[12,13], Bhunia and Maiti[ 14,15 ] and others.

In this paper, a single item inventory model is considered for style/fashionable products (like Winter garments, electronic goods, motor vehicles, computers etc.) whose demand is different (either stock and selling price dependent or constant) at different places of a country. Here time horizon is infinite. There is a main warehouse (MW) with a sales depot in a central place of the country and one Secondary Warehouse (SW) with one sale cum showroom (PW) in each of the $n$-states (here, $n=2$ ). It is assumed that the rent of the showrooms (PW) at the heart of the market place is higher than that rent of the secondary warehouses (SW's) as they are at a distance from market place. As the distance from the market place to secondary warehouse increases, the rent of the SW's decreases but the transportation cost to transport the product from SW to PW increases. The products are transferred from SW's to PW's in bulk or continuous release pattern depending upon the nature of demand at PW's i.e., constant or stocklevel dependent respectively. For the places where showroom-display generates the demand, showrooms are obviously replenished continuously empting the corresponding SW's first. For constant demand also, to maintain the goodwill of the shop, showroom is filled up by bulk units at certain time intervals incurring some additional expenditure due to its high rent than the SW's as SW is emptied first. The problem is divided into two cases: The whole system is assumed to be under a single (case-1) and different management systems (case-2). Here the time periods at different sales counter are assumed to be same and hence there are different scenarios depending upon the stock positions at the showrooms at the end of time period. If an item is not exhausted at a showroom, it is immediately sold at a reduced price. On the other hand, shortages are allowed if the item is exhausted early. To minimize the loss of goodwill, a restriction is imposed on the amount of shortages.
In Case-I, the problem is solved by GRG method for single objective and in case-2, the problem is solved by GRG method using Interactive Satisficing method [Sakawa [16] ] for multi-objectives. Some numerical examples for different scenarios are presented to illustrate the cases.

## 2. Assumptions and notations

The inventory problem is developed under the following assumptions and notations:

## Assumptions:

(i) Rate of replenishment is infinite and the replenishment size is finite.
(ii) The inventory-planning horizon is infinite and the inventory system involves only one item.
(iii) Limited shortages are allowed.
(iv) There is no quantity discount.
(v) The units are sold from show room (i.e., primary warehouses viz., $\mathrm{PW}_{1}, \mathrm{PW}_{2}$ and $\mathrm{PW}_{3}$ ).

## Notations:

(i) $\quad \mathrm{S}_{\mathrm{w}}^{(\mathrm{j})}=$ Initial inventory for the item in the main warehouse (MW) which are sold through $\mathrm{PW}_{\mathrm{j}}$.
(ii) $\mathrm{S}_{\mathrm{j}}=$ Highest level of stock at $\mathrm{SW}_{\mathrm{j}}$.
(iii) $\mathrm{q}_{\mathrm{j}}(\mathrm{t})=$ Inventory level at any time t for the units which are sold through $\mathrm{PW}_{\mathrm{j}}$.
(iv) $\quad \mathrm{K}_{\mathrm{j}}=$ Number of units which are transferred from MW to $\mathrm{SW}_{\mathrm{j}}$ in each shipments.
(v) $\quad \mathrm{K}_{\mathrm{j}}^{\prime}=$ Number of units which are transferred from $\mathrm{SW}_{\mathrm{j}}$ to $\mathrm{PW}_{\mathrm{j}}$ in each shipments.
(vi) $\mathrm{t}_{1}=$ Consumption time of $\mathrm{K}_{1}$ units in $\mathrm{PW}_{1}$.
(vii) $\mathrm{t}_{1}^{\prime}=$ Consumption time of $\mathrm{K}_{2}$ units in $\mathrm{PW}_{2}$.
(viii) $\mathrm{t}_{2}^{\prime}=$ Consumption time of $\mathrm{K}_{2}^{\prime}$ units in $\mathrm{PW}_{2}$.
(ix) $\quad \mathrm{n}_{\mathrm{j}}=$ Number of shipments by which the items are transported from MW to $\mathrm{SW}_{\mathrm{j}}$.
(x) $\quad \mathrm{m}_{2}=$ Number of shipments by which the stock $\mathrm{K}_{2}$ are transported from $\mathrm{SW}_{2}$ to $\mathrm{PW}_{2}$ in $\mathrm{t}_{1}^{\prime}$ time.
(xi) $\quad \mathrm{m}_{3}=$ Number of shipments by which the stock $\mathrm{S}_{2}$ are transported from $\mathrm{SW}_{2}$ to $\mathrm{PW}_{2}$ during $\left[\mathrm{t}_{21}, \mathrm{t}_{22}\right]$.
(xii) $\quad \mathrm{W}_{\mathrm{j}}=$ Highest level of stock at $\mathrm{PW}_{\mathrm{j}}$.
(xiii) $\mathrm{t}_{\mathrm{jl}}=$ Time when inventory level of the item is $\left(\mathrm{S}_{\mathrm{j}}+\mathrm{w}_{\mathrm{j}}\right)$.
(xiv) $\quad \mathrm{t}_{\mathrm{j} 2}=$ Time when inventory level of $\mathrm{SW}_{\mathrm{j}}(\mathrm{j}=1,2)$ is zero.
(xv) $\quad \mathrm{t}_{\mathrm{j} 3}=$ Time when inventory level of $\mathrm{PW}_{\mathrm{j}}$ is zero.
(xvi) $t_{32}=$ Time when there is no item left in main warehouse to transfer in $\mathrm{PW}_{3}$.
(xvii) $\mathrm{T}=$ Total time period of $\mathrm{PW}_{1}, \mathrm{PW}_{2}$ and $\mathrm{PW}_{3}$.
(xviii) $\mathrm{R}_{\mathrm{j}}=$ Shortage amount for $\mathrm{PW}_{\mathrm{j}}$.
(xix) $f\left(p_{1}, q_{1}\right)=\left(\alpha_{1}+\beta_{1} q_{1}\right) p_{1}^{-\gamma_{1}} \quad\left(\alpha_{1}, \beta_{1}, \gamma_{1}>0\right)$ is the demand rate at $P W_{1}$ in no shortage period and $\alpha_{1} \mathrm{p}_{1}^{-\gamma_{1}}$ in shortage period.
(xx) $\quad \mathrm{D}=$ Demand rate at $\mathrm{PW}_{2}$.
(xxi) $f\left(p_{3}, q_{3}\right)=\alpha_{3} q_{3}^{\beta_{3}} p_{3}^{-\gamma_{3}}\left(\alpha_{3}, \beta_{3}, \gamma_{3}>0\right)$ is the demand rate at $P_{3}$ in no shortage period and $\alpha_{3} p_{3}^{-\gamma_{3}}$ in shortage period.
(xxii) $d_{j}=$ Distance from $\mathrm{SW}_{\mathrm{j}}$ to $\mathrm{PW}_{\mathrm{j}}$.
(xxiii) $\mathrm{C}_{1}^{\mathrm{M}}=$ The inventory carrying cost per unit per unit time in MW.
(xxiv) $C_{1}^{p w j}=$ The inventory carrying cost per unit per unit time in $\mathrm{PW}_{\mathrm{j}}$.
(xxv) $C_{1}^{s w j}=C_{1}^{p w j}-F C . d_{j}=$ The inventory carrying cost per unit per unit time in $\mathrm{SW}_{j}$.
(xxvi) $C_{2}^{p w j}=$ Shortage cost per unit per unit time $\mathrm{PW}_{j}$.
(xxvii) $\mathrm{C}_{3 \mathrm{j}}=$ Replenishment cost per cycle at $\mathrm{PW}_{\mathrm{j}}$.
(xxviii) $\mathrm{C}=$ Purchasing cost per unit quantity at MW for integrated model.
(xxix) $\quad C_{j}=$ Purchasing cost per unit quantity at $\mathrm{SW}_{\mathrm{j}}$ for non-integrated model.
( xxx ) $\quad \mathrm{p}_{\mathrm{j}}=\mathrm{v}_{\mathrm{j}} \mathrm{c}, \mathrm{v}_{\mathrm{j}}>1$ be the selling price per unit quantity at $\mathrm{PW}_{\mathrm{j}}$ for integrated model.
(xxxi) $p_{1 j}=v_{1 j} c_{j}, v_{1 j}>1$ be the selling price per unit quantity at $\mathrm{PW}_{j}(\mathrm{j}=1,2)$ for non-integrated model.
(xxxii) $p_{13}=v_{13} c_{3}, v_{13}>1$ be the selling price per unit quantity at $\mathrm{PW}_{3}$ for non-integrated model.
(xxxiii) $\mathrm{C}_{\mathrm{tmj}}=$ Fixed transportation cost for transporting $\mathrm{S}_{\mathrm{w}}^{(\mathrm{j})}$ units from MW to $\mathrm{PW}_{\mathrm{j}}$.
(xxxiv) $\mathrm{C}_{\mathrm{tmj}}^{\prime}=$ Transportation cost per unit for transporting $\left(\mathrm{S}_{\mathrm{j}}+\mathrm{w}_{\mathrm{j}}\right.$ ) units from MW to $\mathrm{SW}_{\mathrm{j}}$ and $\mathrm{PW}_{\mathrm{j}}$. (xxxv) $C_{m}^{s w j}=$ Transportation cost per unit per unit distance for transporting $n_{j} K_{j}$ units from MW to $S W_{j}$. ( xxxvi ) $\mathrm{C}_{\mathrm{swj}}^{\mathrm{pwj}}=$ Transportation cost per unit per unit distance for transporting $\mathrm{S}_{\mathrm{w}}^{(\mathrm{j})}$ units from $\mathrm{SW}_{\mathrm{j}}$ to $\mathrm{PW}_{\mathrm{j}}$.

## 2. Mathematical formulation

### 2.2 2.1 System-I: MW-SW ${ }_{1}$-PW ${ }_{1}$ Distribution system:

Initially, units are transferred from MW to $\mathrm{SW}_{1}$ and $\mathrm{PW}_{1}$ and then sold from $\mathrm{PW}_{1}$ via $\mathrm{SW}_{1}$. Once $\mathrm{K}_{1}$ units are sold from $\mathrm{PW}_{1}$ and these $\mathrm{K}_{1}\left(\mathrm{~K}_{1}<\mathrm{S}_{1}\right)$ units are continuously released from $\mathrm{SW}_{1}$, immediately $\mathrm{K}_{1}$ units are transported from MW to $\mathrm{SW}_{1}$. It is assumed that $\mathrm{n}_{1}$ such transportation are made during the whole time period T .


Fig-1 (Pictorial representation of System-1 for Case-I)
Hence the differential equation governing this system during $(0, T)$ is

$$
\frac{\mathrm{dq}_{1}(\mathrm{t})}{\mathrm{dt}}= \begin{cases}-\left(\alpha_{1}+\beta_{1} \mathrm{~W}_{1}\right) \mathrm{p}_{1}^{-\gamma_{1}}, & 0 \leq \mathrm{t} \leq \mathrm{t}_{12}  \tag{1}\\ -\left(\alpha_{1}+\beta_{1} \mathrm{q}_{1}\right) \mathrm{p}_{1}^{-\gamma_{1}}, & \mathrm{t}_{12} \leq \mathrm{t} \leq \mathrm{t}_{13} \\ -\alpha_{1} \mathrm{p}_{1}^{-\gamma_{1}}, & \mathrm{t}_{13} \leq \mathrm{t} \leq \mathrm{T}\end{cases}
$$

subject to the conditions

$$
\mathrm{q}_{1}(\mathrm{t})= \begin{cases}\mathrm{S}_{\mathrm{w}}^{(1)}, & \mathrm{t}=0  \tag{2}\\ \mathrm{~S}_{\mathrm{W}}^{(1)}-\mathrm{iK}_{1}, & \mathrm{t}=\mathrm{it}_{1} \\ \mathrm{~S}_{1}+\mathrm{W}_{1}, & \mathrm{t}=\mathrm{t}_{11} \\ \mathrm{~W}_{1}, & \mathrm{t}=\mathrm{t}_{12} \\ 0, & \mathrm{t}=\mathrm{t}_{13} \\ -\mathrm{R}_{1}, & \mathrm{t}=\mathrm{T}\end{cases}
$$

Using the conditions, the solutions of the differential equation (1) is given by

$$
\mathrm{q}_{1}(\mathrm{t})= \begin{cases}\mathrm{S}_{\mathrm{W}}^{(1)}-\left(\alpha_{1}+\beta_{1} \mathrm{~W}_{1}\right) \mathrm{p}_{1}^{-\gamma_{1}} . \mathrm{t}, & 0 \leq \mathrm{t} \leq \mathrm{t}_{12}  \tag{3}\\ \frac{\alpha_{1}}{\beta_{1}}\left\{\mathrm{e}^{\left(\mathrm{t}_{13}-\mathrm{t}\right) \beta_{1} p_{1}^{-\gamma_{1}}}-1\right\}, & \mathrm{t}_{12} \leq \mathrm{t} \leq \mathrm{t}_{13} \\ \left(\mathrm{t}_{13}-\mathrm{t}\right) \alpha_{1} \mathrm{p}_{1}^{-\gamma_{1}}, & \mathrm{t}_{13} \leq \mathrm{t} \leq \mathrm{T}\end{cases}
$$

and from (2) we have

$$
\begin{align*}
& \mathrm{t}_{1}=\frac{\mathrm{K}_{1} \mathrm{p}_{1}^{\gamma_{1}}}{\alpha_{1}+\beta_{1} \mathrm{~W}_{1}}, \quad \mathrm{t}_{11}=\mathrm{n}_{1} \mathrm{t}_{1}, \quad \mathrm{t}_{12}=\mathrm{t}_{11}+\frac{\mathrm{S}_{1} \mathrm{p}_{1}^{\gamma_{1}}}{\alpha_{1}+\beta_{1} \mathrm{~W}_{1}}, \quad \mathrm{t}_{13}=\mathrm{t}_{12}+\frac{\mathrm{p}_{1}^{\gamma_{1}}}{\beta_{1}} \log \left|\frac{\alpha_{1}+\beta_{1} \mathrm{~W}_{1}}{\alpha_{1}}\right|  \tag{4}\\
& \mathrm{S}_{\mathrm{w}}^{(1)}=\mathrm{S}_{1}+\mathrm{W}_{1}+\mathrm{n}_{1} \mathrm{~K}_{1}, \quad \mathrm{R}_{1}=\alpha_{1} \mathrm{p}_{1}^{-\gamma_{1}}\left(\mathrm{t}_{13}-\mathrm{t}\right) \tag{5}
\end{align*}
$$

$\mathrm{C}_{\mathrm{HOL}}^{\mathrm{M}_{1}}=$ Holding cost for the units in the main warehouse, which are to be transferred in the $\mathrm{SW}_{1}$

$$
\begin{equation*}
=C_{1}^{\mathrm{M}}\left[\mathrm{n}_{1} \mathrm{~K}_{1} \mathrm{t}_{1}+\left(\mathrm{n}_{1}-1\right) \mathrm{K}_{1} \mathrm{t}_{1}+\left(\mathrm{n}_{1}-2\right) \mathrm{K}_{1} \mathrm{t}_{1}+\ldots \ldots \ldots .+2 \mathrm{~K}_{1} \mathrm{t}_{1}+\mathrm{K}_{1} \mathrm{t}_{1}\right]=\frac{1}{2} C_{1}^{\mathrm{M}} \mathrm{n}_{1}\left(\mathrm{n}_{1}+1\right) \mathrm{K}_{1} \mathrm{t}_{1} \tag{6}
\end{equation*}
$$

$\mathrm{C}_{\mathrm{HOL}}^{\mathrm{SW} 1}=$ Holding cost for the units in $\mathrm{SW}_{1}$

$$
\begin{gather*}
=C_{1}^{s w 1}\left[n_{1}\left\{\int_{0}^{t_{1}} q_{1}(t) d t-\left(S_{w}^{1}-K_{1}\right) t_{1}\right\}+n_{1}\left(S_{1}-K_{1}\right) t_{1}+\int_{t_{11}}^{t_{12}}\left\{q_{1}(t)-W_{1}\right\} d t\right]  \tag{7}\\
=C_{1}^{S W 1}\left[n_{1}\left\{S_{1} t_{1}-\frac{1}{2}\left(\alpha_{1}+\beta_{1} W_{1}\right) p_{1}^{-\gamma_{1}} t_{1}^{2}\right\}+\left\{S_{w}^{1}\left(t_{12}-t_{11}\right)-\frac{1}{2}\left(\alpha_{1}+\beta_{1} W_{1}\right) p_{1}^{-\gamma_{1}}\left(t_{12}^{2}-t_{11}^{2}\right)-W_{1}\left(\mathrm{t}_{12}-t_{11}\right)\right\}\right]
\end{gather*}
$$

$\mathrm{C}_{\mathrm{S}}^{\mathrm{W} 1}=$ Shortage cost in $\mathrm{PW}_{1}=\frac{1}{2} \mathrm{C}_{2}^{\mathrm{w} 1} \alpha_{1} \mathrm{p}_{1}^{-\gamma_{1}}\left(\mathrm{~T}-\mathrm{t}_{13}\right)^{2}$
$\mathrm{TC}^{\mathrm{M} 1}=$ Total transportation cost for transporting $\mathrm{S}_{\mathrm{w}}^{(1)}$ units from main warehouse (MW) to $\mathrm{SW}_{1}$ and $\mathrm{PW}_{1}$

$$
\begin{equation*}
=\mathrm{C}_{\mathrm{tml}}+\mathrm{C}_{\mathrm{tm} 1}^{\prime}\left(\mathrm{S}_{1}+\mathrm{W}_{1}\right)+\mathrm{n}_{1} \mathrm{~K}_{1} \mathrm{C}_{\mathrm{ml}}^{\mathrm{sw} 1}+\mathrm{C}_{\mathrm{sw} 1}^{\mathrm{pw} 1}\left(\mathrm{~S}_{\mathrm{w}}^{(1)}-\mathrm{W}_{1}\right) \mathrm{d}_{1} \tag{9}
\end{equation*}
$$

### 2.3 System-II: MW-SW $\mathbf{2}_{2}-\mathrm{PW}_{2}$ Distribution system:

Initially, units are transferred from MW to $\mathrm{SW}_{2}$ and $\mathrm{PW}_{2}$ and then sold from $\mathrm{PW}_{2}$ via $\mathrm{SW}_{2}$. Once $\mathrm{k}_{2}$ units are sold from $\mathrm{PW}_{2}$, immediately these $\mathrm{k}_{2}^{\prime}\left(\mathrm{k}_{2}^{\prime}<\mathrm{K}_{2}<\mathrm{S}_{2}\right)$ units are transported from $\mathrm{SW}_{2}$ to $\mathrm{PW}_{2}$. And after $\mathrm{K}_{2}\left(=\mathrm{m}_{3} . \mathrm{k}_{2}^{\prime}\right)$ units are sold from $\mathrm{PW}_{2}$, immediately these $\mathrm{K}_{2}$ units are transported from MW to $\mathrm{SW}_{2}$. It is assumed that $\mathrm{n}_{2}$ such transportation (from MW to $\mathrm{SW}_{2}$ ) are made during the whole time period T .


Fig-2 (Pictorial representation of System-2 for Case-I)
Hence the differential equation governing this system during $(0, T)$ is

$$
\frac{\mathrm{dq}_{2}(\mathrm{t})}{\mathrm{dt}}= \begin{cases}-\mathrm{D}, & 0 \leq \mathrm{t} \leq \mathrm{t}_{23}  \tag{10}\\ -\delta \mathrm{D}, & \mathrm{t}_{23} \leq \mathrm{t} \leq \mathrm{T}\end{cases}
$$

subject to the conditions
$q_{1}(t)= \begin{cases}S_{w}^{(2)}, & t=0 \\ S_{w}^{(2)}-i K_{2}^{\prime}, & t=i t_{2}^{\prime}\left(i=1,2, \ldots ., m_{2}\right) \\ S_{w}^{(2)}-j K_{2}, & t=j t_{1}^{\prime}\left(j=1,2, \ldots, n_{2}\right) \\ S_{2}+W_{2}, & t=t_{21} \\ S_{2}+W_{2}-K_{2}^{\prime}, & t=t_{21}+t_{2} . \\ W_{2}, & t=t_{22} \\ 0, & t=t_{23} \\ -R_{2}, & t=T\end{cases}$
Using the conditions (11), the solutions of the differential equation (10) is given by
$\mathrm{q}_{2}(\mathrm{t})=\left\{\begin{array}{ccc}\mathrm{S}_{\mathrm{W}}^{(2)}-\mathrm{Dt} & , & 0 \leq \mathrm{t} \leq \mathrm{t}_{23} \\ \delta \mathrm{D}\left(\mathrm{T}-\mathrm{t}_{23}\right) & , & \mathrm{t}_{23} \leq \mathrm{t} \leq \mathrm{T}\end{array}\right.$
Also from (12), we have
$\mathrm{K}_{2}^{\prime}=\mathrm{Dt}_{2}{ }^{\prime}, \mathrm{K}_{2}=\mathrm{m}_{2} \mathrm{~K}_{2}^{\prime}, \mathrm{S}_{2}=\mathrm{m}_{3} \mathrm{~K}_{2}^{\prime}, \mathrm{S}_{\mathrm{w}}^{(2)}=\mathrm{S}_{2}+\mathrm{W}_{2}+\mathrm{n}_{2} \mathrm{~K}_{2}$
$t_{1}^{\prime}=m_{2} t_{2}^{\prime}, t_{21}=n_{2} t_{1}^{\prime}, t_{21}=n_{2} t_{1}^{\prime}, t_{22}=t_{21}+m_{3} t_{2}^{\prime}, t_{23}=t_{22}+\frac{W_{2}}{D}$
$\mathrm{C}_{\mathrm{HOL}}^{\mathrm{M}_{2}}=$ Holding cost for the units in the main warehouse, which are to be transferred in the $\mathrm{SW}_{2}$

$$
\begin{equation*}
=\mathrm{C}_{1}^{\mathrm{M}}\left[\mathrm{n}_{2} \mathrm{~K}_{2} \mathrm{t}_{1}^{\prime}+\left(\mathrm{n}_{2}-1\right) \mathrm{K}_{2} \mathrm{t}_{1}^{\prime}+\left(\mathrm{n}_{2}-2\right) \mathrm{K}_{2} \mathrm{t}_{1}+\ldots \ldots .+2 \mathrm{~K}_{2} \mathrm{t}_{1}+\dot{K}_{2} \mathrm{t}_{1}\right]=\frac{1}{2} \mathrm{C}_{1}^{\mathrm{M}} \mathrm{n}_{2}\left(\mathrm{n}_{2}+1\right) \mathrm{K}_{2} \mathrm{t}_{1}^{\prime} \tag{15}
\end{equation*}
$$

$\mathrm{C}_{\mathrm{HOL}}^{\mathrm{sw} 2}=$ Holding cost for the units in $\mathrm{SW}_{2}$

$$
\begin{align*}
&=C_{1}^{s w 2} {\left[n_{2}\left\{m_{2} \mathrm{~K}_{2}^{\prime} \mathrm{t}_{2}^{\prime}+\left(\mathrm{m}_{2}-1\right) \mathrm{K}_{2} \mathrm{t}_{2}^{\prime}++^{\prime}\left(\mathrm{m}_{2}-2\right) \mathrm{K}_{2} \mathrm{t}_{2}+\ldots \ldots \ldots .+2 \mathrm{~K}_{2} \mathrm{t}_{2}+\dot{K}_{2}^{\prime} \mathrm{t}_{2}^{\prime}\right\}\right.} \\
&\left.+\left\{\mathrm{m}_{3} \mathrm{~K}_{2}^{\prime} \mathrm{t}_{2}^{\prime}+\left(\mathrm{m}_{3}-1\right) \mathrm{K}_{2} \mathrm{t}_{2}^{\prime}+\left(\mathrm{m}_{3}-2\right) \mathrm{K}_{2} \mathrm{t}_{2}+\ldots \ldots \ldots .+2 \mathrm{~K}_{2} \mathrm{t}_{2}+\dot{K}_{2} \mathrm{t}_{2}^{\prime}\right\}\right] \\
&=\frac{1}{2} \mathrm{C}_{1}^{\mathrm{sw} 2} \mathrm{~K}_{2}^{\prime} \mathrm{t}_{2}^{\prime}\left[\mathrm{n}_{2} \mathrm{~m}_{2}\left(\mathrm{~m}_{2}+1\right)+\mathrm{m}_{3}\left(\mathrm{~m}_{3}+1\right)\right] \tag{16}
\end{align*}
$$

$\mathrm{C}_{\mathrm{HOL}}^{\mathrm{pw} 2}=$ Holding cost for the units in $\mathrm{PW}_{2}$

$$
\begin{align*}
= & C_{1}^{\mathrm{pw} 2}\left[\left(\mathrm{~W}_{2}-\mathrm{K}_{2}^{\prime}\right) \mathrm{t}_{22}+\left\{\int_{0}^{\mathrm{t}_{2}^{\prime}} \mathrm{q}_{2}(\mathrm{t}) \mathrm{dt}-\left(\mathrm{S}_{\mathrm{w}}^{2}-\mathrm{K}_{2}\right) \mathrm{t}_{2}\right\}^{\prime}+\int_{\mathrm{t}_{22}}^{\mathrm{t}_{23}} \mathrm{q}_{2}(\mathrm{t}) \mathrm{dt}\right] \\
& =\mathrm{C}_{1}^{\mathrm{pw} 2}\left[\left(\mathrm{~W}_{2}-\mathrm{K}_{2}^{\prime}\right) \mathrm{t}_{22}-\frac{\mathrm{D}}{2} \mathrm{t}_{2}{ }^{2}+\mathrm{K}_{2} \mathrm{t}_{2}^{\prime}+\left\{\mathrm{S}_{\mathrm{w}}^{2}\left(\mathrm{t}_{23}-\mathrm{t}_{22}\right)-\frac{\mathrm{D}}{2}\left(\mathrm{t}_{23}-\mathrm{t}_{22}\right)\right\}\right]^{2}  \tag{17}\\
\mathrm{C}_{\mathrm{S}}^{\mathrm{pw} 2}= & \text { Shortage cost in } \mathrm{PW}_{2}=\frac{\delta \mathrm{D}}{2} \mathrm{C}_{2}^{\mathrm{pw} 2}\left(\mathrm{~T}-\mathrm{t}_{23}\right)^{2} \tag{18}
\end{align*}
$$

$\mathrm{TC}{ }^{\mathrm{M} 2}=$ Total transportation cost for transporting $\mathrm{S}_{\mathrm{w}}^{2}$ units from main warehouse (MW) to $\mathrm{SW}_{2}$ and $\mathrm{PW}_{2}$

$$
\begin{equation*}
=\mathrm{C}_{\mathrm{tm} 2}+\mathrm{C}_{\mathrm{tm} 2}^{\prime}\left(\mathrm{S}_{2}+\mathrm{W}_{2}\right)+\mathrm{n}_{2} \mathrm{~K}_{2} \mathrm{C}_{\mathrm{m}}{ }^{\mathrm{sw} 2}+\left(\mathrm{n}_{2} \mathrm{~m}_{2}+\mathrm{m}_{3}\right) \mathrm{K}_{2}{ }^{\prime} \mathrm{d}_{2} \mathrm{C}_{\mathrm{sw} 2}^{\mathrm{pw} 2} \tag{19}
\end{equation*}
$$

### 2.4 System-III: MW- PW 3 Distribution system:

The units are sold at the showroom $\mathrm{PW}_{3}$ and are continuously transferred from MW.


Fig-3 (Pictorial representation of System-3 for Case-I)
Hence the differential equation governing this system during $(0, T)$ is
$\frac{\mathrm{dq}_{1}(\mathrm{t})}{\mathrm{dt}}= \begin{cases}-\alpha_{3} \mathrm{~W}_{3}^{\beta_{3}} \mathrm{p}_{3}^{-\gamma_{3}}, & 0 \leq \mathrm{t} \leq \mathrm{t}_{32} \\ -\alpha_{3} q_{3}^{\beta_{3}} p_{3}^{-\gamma_{3}}, & \mathrm{t}_{32} \leq \mathrm{t} \leq \mathrm{t}_{33} \\ -\alpha_{3} \mathrm{p}_{3}^{-\gamma_{3}}, & \mathrm{t}_{33} \leq \mathrm{t} \leq \mathrm{T}\end{cases}$
subject to the conditions
$\mathrm{q}_{3}(\mathrm{t})=\left\{\begin{array}{lll}\mathrm{S}_{\mathrm{w}}^{(3)} & , & \mathrm{t}=0 \\ \mathrm{~W}_{3} & , & \mathrm{t}=\mathrm{t}_{32} \\ 0 & , & \mathrm{t}=\mathrm{t}_{33} \\ -\mathrm{R}_{3} & , & \mathrm{t}=\mathrm{T}\end{array}\right.$
Using the conditions, the solutions of the differential equation (22) is given by
$\mathrm{q}_{3}(\mathrm{t})=\left\{\begin{array}{lll}\mathrm{S}_{\mathrm{W}}^{(3)}-\alpha_{3} \mathrm{~W}_{3}^{\beta_{3}} \mathrm{p}_{3}^{-\gamma_{3}} \cdot \mathrm{t} & 0 \leq \mathrm{t} \leq \mathrm{t}_{32} \\ \left\{\alpha_{3}\left(1-\beta_{3}\right) \mathrm{p}_{3}^{-\gamma_{3}} \cdot\left(\mathrm{t}_{33}-\mathrm{t}\right)\right\}^{\left(1-\beta_{3}\right)} & , & \mathrm{t}_{32} \leq \mathrm{t} \leq \mathrm{t}_{33} \\ \alpha_{3} \mathrm{p}_{3}^{-\gamma_{3}} \cdot\left(\mathrm{t}_{33}-\mathrm{t}\right) & , & \mathrm{t}_{33} \leq \mathrm{t} \leq \mathrm{T}\end{array}\right.$
Also from (23), we have
$S_{w}^{(3)}=S_{3}+W_{3}, \quad R_{3}=\alpha_{3}\left(T-t_{33}\right) p_{3}^{-\gamma_{3}}$
$\mathrm{t}_{32}=\frac{\mathrm{S}_{\mathrm{w}}^{(3)}-\mathrm{W}_{3}}{\alpha_{3} \mathrm{~W}_{3}^{\beta_{3}} \mathrm{p}_{3}^{-\gamma_{3}}}, \quad \mathrm{t}_{33}=\mathrm{t}_{32}+\frac{\mathrm{p}_{3}^{\gamma_{3}} \mathrm{~W}_{3}{ }^{1-\beta_{3}}}{\alpha_{3}\left(1-\beta_{3}\right)}$
$\mathrm{C}_{\mathrm{HOL}}^{\mathrm{M}_{3}}=$ Holding cost for the units in the main warehouse, which are to be transferred in the $\mathrm{PW}_{2}$

$$
\begin{equation*}
=C_{1}^{\mathrm{M}}\left[\int_{0}^{\mathrm{t}_{32}} \mathrm{q}_{3}(\mathrm{t}) \mathrm{dt}-\mathrm{W}_{3} \mathrm{t}_{32}\right]=\mathrm{C}_{1}^{\mathrm{M}}\left[\left(\mathrm{~S}_{\mathrm{w}}^{(3)}-\mathrm{W}_{3}\right) \mathrm{t}_{32}-\frac{1}{2} \alpha_{3} \mathrm{~W}_{3}^{\beta_{3}} \mathrm{p}_{3}^{-\gamma_{3}} \mathrm{t}_{32}^{2}\right] \tag{25}
\end{equation*}
$$

$\mathrm{C}_{\mathrm{HOL}}^{\mathrm{pw} 3}=$ Holding cost for the units in $\mathrm{PW}_{3}$
$=C_{1}^{\text {pw } 2}\left[W_{3} t_{32}+\left\{\int_{t_{32}}^{t_{33}} q_{3}(t) d t\right]=C_{1}^{\text {pw } 3}\left[W_{3} t_{32}+\frac{1-\beta_{3}}{2-\beta_{3}}\left\{\alpha_{3}\left(1-\beta_{3}\right) p_{3}^{-\gamma_{3}}\right\}^{1-\beta_{3}}\left(t_{33}-t_{32}\right)\right]\right.$
$\mathrm{C}_{\mathrm{S}}^{\mathrm{pw} 3}=-\mathrm{C}_{23} \int_{\mathrm{t}_{33}}^{\mathrm{t}_{34}} \mathrm{q}(\mathrm{t}) \mathrm{dt}, \quad \mathrm{R}_{3}=\frac{\mathrm{C}_{23}}{2} \alpha_{3} \mathrm{p}_{3}^{-\gamma_{3}}\left(\mathrm{~T}-\mathrm{t}_{33}\right)^{2}$
$T C^{\mathrm{M} 3}=$ Total transportation cost for transporting $\mathrm{S}_{\mathrm{w}}^{3}$ units from main warehouse (MW) to $\mathrm{PW}_{3}$ $=\mathrm{C}_{\mathrm{tm} 3}+\mathrm{S}_{\mathrm{w}}^{(3)} \mathrm{C}_{\mathrm{tm} 3}^{\mathrm{pw} 3}$
where $\mathrm{C}_{\mathrm{tm} 3}=$ Fixed transportation cost for transporting $\mathrm{S}_{\mathrm{w}}^{3}$ units from MW to $\mathrm{PW}_{2}$
$\mathrm{C}_{\mathrm{sw} 3}^{\mathrm{pw} 3}=$ Transportation cost per unit for transporting $\mathrm{S}_{\mathrm{w}}^{3}$ units from MW to $\mathrm{PW}_{3}$.

## Case-I: When the entire system is under single a management.

In this case, it is assumed that each warehouses and showrooms are owned by a single business house.


Hence, the problem is to Maximize PROF
where PROF $=\frac{1}{\mathrm{~T}}\left(\mathrm{PROF}_{1}+\mathrm{PROF}_{2}+\mathrm{PROF}_{3}\right)$
and $\mathrm{PROF}_{1}=$ Total profit for the system- $=\left[\left(\mathrm{p}_{1}-\mathrm{c}\right) \mathrm{S}_{\mathrm{w}}^{1}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{M1}}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{pw} 1}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{sw} 1}-\mathrm{C}_{\mathrm{S}}^{\mathrm{pw} 1}-\mathrm{C}_{31}-\mathrm{TC}^{\mathrm{M1}}\right]$
$\mathrm{PROF}_{2}=$ Total profit for the system-II $=\left[\left(\mathrm{p}_{2}-\mathrm{c}\right) \mathrm{S}_{\mathrm{w}}^{2}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{M} 2}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{pw} 2}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{sw} 2}-\mathrm{C}_{\mathrm{S}}^{\mathrm{pw} 2}-\mathrm{C}_{32}-\mathrm{TC}^{\mathrm{M} 2}\right]$
$\mathrm{PROF}_{3}=$ Total profit for the system-I $=\left[\left(p_{3}-c\right) S_{\mathrm{w}}^{3}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{M3}}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{pw} 3}-\mathrm{C}_{\mathrm{S}}^{\mathrm{pw} 3}-\mathrm{C}_{33}-\mathrm{TC}^{\mathrm{M3}}\right]$.
Case-II: When the system is a non-integrated one i.e., different parts of the distribution system are under separate managements.
In this case it is assumed that there are three separate managements (shown in diagram).


In this case our problem is a multi-objective problem where objective functions are
Maximize $\left\{\mathrm{PF}_{1}, \mathrm{PF}_{2}, \mathrm{PF}_{3}\right\}$
Subject to the condition $S=S_{w}^{(1)}+S_{w}^{(2)}+S_{w}^{(3)}$ (where $S$ is the total capacity of MW).Where
$\mathrm{PF}_{1}=$ Average profit for the system-I $=\left[\left(\mathrm{p}_{11}-\mathrm{c}_{1}\right) \mathrm{S}_{\mathrm{w}}^{(1)}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{pw1}}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{sw1}}-\mathrm{C}_{\mathrm{S}}^{\mathrm{pw} 1}-\mathrm{C}_{31}-\mathrm{TC}^{\mathrm{M1}}\right] / \mathrm{T}$
$\mathrm{PF}_{2}=$ Total profit for the system-II $=\left[\left(\mathrm{p}_{12}-\mathrm{c}_{2}\right) \mathrm{S}_{\mathrm{w}}^{(2)}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{pw} 2}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{sw} 2}-\mathrm{C}_{\mathrm{S}}^{\mathrm{pw} 2}-\mathrm{C}_{32}-\mathrm{TC}^{\mathrm{M} 2}\right] / \mathrm{T}$
$\mathrm{PF}_{3}=$ Total profit for the system-III

$$
=\left[\left(c_{1}-c\right) S_{\mathrm{w}}^{(1)}+\left(\mathrm{c}_{2}-\mathrm{c}\right) \mathrm{S}_{\mathrm{w}}^{(2)}+\left(\mathrm{p}_{13}-\mathrm{c}\right) \mathrm{S}_{\mathrm{w}}^{(3)}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{M1}}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{M} 2}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{M} 3}-\mathrm{C}_{\mathrm{HOL}}^{\mathrm{pw} 3}-\mathrm{C}_{\mathrm{s}}^{\mathrm{pw} 3}-\mathrm{C}_{33}-\mathrm{TC}^{\mathrm{M} 3}\right] / \mathrm{T} .
$$

## 6. Interactive Approach

Now considering the imprecise nature of DM's judgment, DM may have different fuzzy or imprecise goals for each of the objective functions and hence interactive approach is used for the man-machine interaction.
To derive the membership functions $\mu_{\mathrm{PF}_{1}}, \mu_{\mathrm{PF}_{2}}, \mu_{\mathrm{PF}_{3}}$ for the corresponding objective functions $\mathrm{PF}_{1}$, $\mathrm{PF}_{2}$ and $\mathrm{PF}_{3}$ respectively from $\mathrm{DM}^{\prime} \mathrm{s}$ viewpoint, we first calculate individual minimum ( i.e. $\mathrm{PF}_{1}^{\min }, \mathrm{PF}_{2}^{\min }, \mathrm{PF}_{3}^{\min }$ ) and individual maximum ( i.e. $\mathrm{PF}_{1}^{\max }, \mathrm{PF}_{2}^{\max }, \mathrm{PF}_{3}^{\max }$ ) by a non-linear optimization technique.
With the help of individual minimum and maximum, the DM can select any one from among the following three types of membership functions
i) Linear membership functions
ii) Quadratic membership functions
iii) Exponential membership functions.

The membership functions $\mu_{\mathrm{PF}_{1}}, \quad \mu_{\mathrm{PF}_{2}}$ and $\mu_{\mathrm{PF}_{3}}$ may be written as

$$
\mu_{\mathrm{PF}_{\mathrm{K}}}= \begin{cases}0 & \text { if }  \tag{31}\\ \mathrm{PF}_{K} \leq \mathrm{PF}_{\mathrm{K}}^{0} \\ d_{K} & \text { if } \quad \mathrm{PF}_{\mathrm{K}}^{0} \leq \mathrm{PF}_{\mathrm{K}} \leq \mathrm{PF}_{\mathrm{K}}^{1} \\ 0 & \text { if } \\ \mathrm{PF}_{\mathrm{K}} \geq \mathrm{PF}_{\mathrm{K}}^{1}\end{cases}
$$

where $\mathrm{PF}_{\mathrm{K}}^{1}$ and $\mathrm{PF}_{\mathrm{K}}^{0}$ are to be chosen such that $\mathrm{PF}_{\mathrm{K}}^{\min } \leq \mathrm{PF}_{\mathrm{K}}^{1} \leq \mathrm{PF}_{\mathrm{K}}^{0} \leq \mathrm{PF}_{\mathrm{K}}^{\max }$ and $\mathrm{d}_{\mathrm{K}}$ is a strictly monotonic decreasing continuous function of $\mathrm{PF}_{\mathrm{K}}$ which may be linear or non-linear.

### 6.1 Linear membership function (Type-I)

For each objective function, the corresponding linear membership functions may be as follows:

$$
\mu_{\mathrm{PF}_{\mathrm{K}}}=\left\{\begin{array}{l}
0 \quad \text { if } \quad \mathrm{PF}_{\mathrm{K}} \leq \mathrm{PF}_{\mathrm{K}}^{0}  \tag{32}\\
1-\frac{\mathrm{PF}_{\mathrm{K}}^{1}-\mathrm{PF}_{\mathrm{K}}}{\mathrm{P}_{\mathrm{K}}} \text { if } \mathrm{PF}_{\mathrm{K}}^{0} \leq \mathrm{PF}_{\mathrm{K}} \leq \mathrm{PF}_{\mathrm{K}}^{1} \\
1
\end{array} \text { if } \mathrm{PF}_{\mathrm{K}} \geq \mathrm{PF}_{\mathrm{K}}^{1} \mathrm{l}\right.
$$

where $\mathrm{PF}_{\mathrm{K}}^{0}$ and $\mathrm{PF}_{\mathrm{K}}^{1}$ are to be chosen such that $\mathrm{PF}_{\mathrm{K}}^{\min } \leq \mathrm{PF}_{\mathrm{K}}^{0} \leq \mathrm{PF}_{\mathrm{K}}^{1} \leq \mathrm{PF}_{\mathrm{K}}^{\max }$ and $\mathrm{P}_{\mathrm{K}}=\mathrm{PF}_{\mathrm{K}}^{1}-\mathrm{PF}_{\mathrm{K}}^{0}$ is the tolerance of $k$-th objective function $\mathrm{PF}_{\mathrm{K}}$.

### 6.2. Quadratic membership function (Type-II)

For each of the objective functions, the corresponding quadratic membership functions may be

$$
\mu_{\mathrm{PF}_{\mathrm{K}}}=\left\{\begin{array}{l}
0 \quad \text { if } \quad \mathrm{PF}_{\mathrm{K}} \leq \mathrm{PF}_{\mathrm{K}}^{1}  \tag{33}\\
1-\left(\frac{\mathrm{PF}_{\mathrm{K}}^{1}-\mathrm{PF}_{\mathrm{K}}}{\mathrm{P}_{\mathrm{K}}}\right)^{2} \text { if } \quad \mathrm{PF}_{\mathrm{K}}^{0} \leq \mathrm{PF}_{\mathrm{K}} \leq \mathrm{PF}_{\mathrm{K}}^{1} \\
1 \quad \text { if } \quad \mathrm{PF}_{\mathrm{K}} \geq \mathrm{PF}_{\mathrm{K}}^{1}
\end{array}\right.
$$

where $\mathrm{PF}_{\mathrm{K}}^{1}$ and $\mathrm{PF}_{\mathrm{K}}^{0}$ are to be chosen such that $\mathrm{PF}_{\mathrm{K}}^{\min } \leq \mathrm{PF}_{\mathrm{K}}^{0} \leq \mathrm{PF}_{\mathrm{K}}^{1} \leq \mathrm{PF}_{\mathrm{K}}^{\max }$ and $\mathrm{P}_{\mathrm{K}}=\mathrm{PF}_{\mathrm{K}}^{1}-\mathrm{PF}_{\mathrm{K}}^{0}$ is the tolerance of k -th objective function $\mathrm{PF}_{\mathrm{K}}$.


Fig.-4 (Pictorial representation of linear $\mu_{\mathrm{PF}_{\mathrm{K}}}$ )


Fig.-5 (Pictorial representation of quadratic $\mu_{\mathrm{PF}_{\mathrm{K}}}$ )

### 6.3. Exponential membership function (Type-III)

For each of the objective functions, the corresponding exponential membership function may be

The constants $\alpha_{\mathrm{K}}>1, \quad \beta_{\mathrm{K}}>0$ can be determined by asking the DM to specify the three points $\mathrm{PF}_{\mathrm{K}}^{1}$, $\mathrm{PF}_{\mathrm{K}}^{0.5}$ and $\mathrm{PF}_{\mathrm{K}}^{0}$ such that $\mathrm{PF}_{\mathrm{K}}^{\min } \leq \mathrm{PF}_{\mathrm{K}}^{0} \leq \mathrm{PF}_{\mathrm{K}}^{0.5} \leq \mathrm{PF}_{\mathrm{K}}^{1} \leq \mathrm{PF}_{\mathrm{K}}^{\max }$ and $\mathrm{P}_{\mathrm{K}}=\mathrm{PF}_{\mathrm{K}}^{1}-\mathrm{PF}_{\mathrm{K}}^{0}$ is the tolerance of k-th objective function $\mathrm{PF}_{\mathrm{K}}$.


Fig.-6 (Pictorial representation of exponential $\mu_{\mathrm{PF}_{\mathrm{K}}}$ )

### 6.4 Fuzzy Satisficing Method:

With the help of two different types of membership functions given by (32) and (33), following Bellman and Zadeh [17] and Zimmermann [18], the given problem (30) can be formulated for a particular choice of DM as

$$
\operatorname{Min} \quad \lambda
$$

$$
\begin{array}{ll}
\text { subject to } & \bar{\mu}_{\mathrm{PF}_{1}}-\mu_{\mathrm{PF}_{1}} \leq \lambda, \quad \mu_{\mathrm{PF}_{1}} \in \text { Type - II } \\
& \bar{\mu}_{\mathrm{PF}_{2}}-\mu_{\mathrm{PF}_{2}} \leq \lambda, \quad \mu_{\mathrm{PF}_{2}} \in \text { Type - I }  \tag{35}\\
& \bar{\mu}_{\mathrm{PF}_{3}}-\mu_{\mathrm{PF}_{3}} \leq \lambda, \quad \mu_{\mathrm{PF}_{3}} \in \text { Type - II }
\end{array}
$$

where $\bar{\mu}_{\mathrm{PF}_{1}}, \bar{\mu}_{\mathrm{PF}_{2}}, \bar{\mu}_{\mathrm{PF}_{3}}$ are the reference values of the membership functions corresponding to the objective functions $\mathrm{PF}_{1}, \mathrm{PF}_{2}, \mathrm{PF}_{3}$.

Here DM selects the above membership functions for the corresponding objective functions and assigns the values of $\bar{\mu}_{\mathrm{PF}_{1}}, \bar{\mu}_{\mathrm{PF}_{2}}, \bar{\mu}_{\mathrm{PF}_{3}}$. Then the above problem can be solved by a non-linear optimization technique and the optimal value of $\lambda$ (say $\lambda^{*}$ ), $\mu_{\mathrm{PF}_{1}}, \mu_{\mathrm{PF}_{2}}, \mu_{\mathrm{PF}_{3}}$ are obtained.
Now after obtaining $\lambda^{*}$, the DM selects the most important objective function from among the objective functions $\mathrm{PF}_{1}, \mathrm{PF}_{2}$ and $\mathrm{PF}_{3}$. Here $\mathrm{PF}_{3}$ (say) is selected and as DM would like to maximize his / her worst case. Then the problem becomes (for $\lambda=\lambda^{*}$ )

$$
\begin{gather*}
\text { Max } \mathrm{PF}_{3}  \tag{36}\\
\text { where } \quad \text { to } \mathrm{PF}_{1} \geq \mathrm{m}_{1}, \quad \mathrm{PF}_{2} \geq \mathrm{m}_{2}, \mathrm{PF}_{3} \geq \mathrm{m}_{3} \\
\mathrm{~m}_{1}=\mathrm{PF}_{1}^{1}-\mathrm{P}_{1}\left(1-\lambda^{*}\right), \quad \text { if the } \mathrm{MF} \text { of first objective } \in \text { Type-II. } \\
\mathrm{m}_{2}=\mathrm{PF}_{2}^{1}-\mathrm{P}_{2} \sqrt{1-\lambda^{*}}, \quad \text { if the MF of second objective } \in \text { Type-I. }  \tag{37}\\
\mathrm{m}_{3}=\mathrm{PF}_{3}^{1}-\mathrm{P}_{3}\left(1-\lambda^{*}\right), \quad \text { if the MF of third objective } \in \text { Type-II. }
\end{gather*}
$$

### 6.5 Pareto Optimal solution

Now, after deriving the optimum decision variables, pareto optimality test is performed following to Sakawa[16]. Let the optimum values, $\mathrm{PF}_{1}^{*}, \mathrm{PF}_{2}^{*}$ and $\mathrm{PF}_{3}^{*}$ are obtained from (37). With these values, the following problem is solving using a non-linear optimization technique.

$$
\begin{aligned}
& \text { Minimize } \mathrm{V}=\left(\varepsilon_{1}+\varepsilon_{2}+\varepsilon_{3}\right) \\
& \text { subject to } \mathrm{PF}_{1}+\varepsilon_{1}=\mathrm{PF}_{1}^{*}, \quad \mathrm{PF}_{2}+\varepsilon_{2}=\mathrm{PF}_{2}^{*}, \quad \mathrm{PF}_{3}+\varepsilon_{3}=\mathrm{PF}_{3}^{*} \\
& \varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3} \geq 0
\end{aligned}
$$

The optimal solutions of (37), say, $\overline{\mathrm{PF}_{1}}, \overline{\mathrm{PF}_{2}}$ and $\overline{\mathrm{PF}_{3}}$ are called the strong Pareto optimal solutions of the problem (30) provided V is very small, otherwise it is weak Pareto Optimum.
This Pareto optimal solution is obtained with a particular set of membership functions for the objectives (say, type-I for the first objective, type-II for the second objective and type-I for the third objective) and their corresponding reference membership values (say, $\bar{\mu}_{\mathrm{PF}_{1}}, \bar{\mu}_{\mathrm{PF}_{2}}, \bar{\mu}_{\mathrm{PF}_{3}}$ ). If DM is not satisfied with the present optimum result, he / she can again repeat the same analysis with different set of membership functions and/or updating the current reference membership values $\bar{\mu}_{\mathrm{PF}_{1}}, \bar{\mu}_{\mathrm{PF}_{2}}, \bar{\mu}_{\mathrm{PF}_{3}}$ to the new membership values taking the current values of the membership functions into consideration. This process may be continued till DM is satisfied with the result, which can be implemented/achieved. This gives an interaction with DM and the machine.

## 7. Numerical Results

Depending upon the occurrence of shortages in $\mathrm{PW}_{1}, \mathrm{PW}_{2}$ and $\mathrm{PW}_{3}$, there may be two scenarios:
Scenario-I : When shortages are allowed in all primary warehouses $\mathrm{PW}_{1}, \mathrm{PW}_{2}$ and $\mathrm{PW}_{3}$.
Scenario-II : When shortages are not allowed in any primary warehouses $\mathrm{PW}_{1}, \mathrm{PW}_{2}$ and $\mathrm{PW}_{3}$.
The above scenarios are illustrated by the input data's given in Table-I and optimal results are given in
Table-II for case-I and for case-II, the results are given through Table-III to VIII.

Table-1(input data)

| $c$ | 10 |
| :---: | :---: |
| $\mathrm{v}_{1}$ | 1.600 |
| $\mathrm{v}_{2}$ | 1.452 |
| $\mathrm{v}_{3}$ | 1.690 |
| $\alpha_{1}$ | 195 |
| $\beta_{1}$ | 0.790 |
| $\gamma_{1}$ | 0.359 |
| D | 210 |
| $\mathrm{C}_{1}$ | 11.5 |


| $\alpha_{3}$ | 110 |
| :---: | :---: |
| $\beta_{3}$ | 0.650 |
| $\gamma_{3}$ | 0.750 |
| $W_{1}$ | 75.79 |
| $W_{2}$ | 48 |
| $W_{3}$ | 85 |
| $d_{1}$ | 1.24 |
| $d_{2}$ | 1.72 |
| $C_{2}$ | 11.5 |


| $\mathrm{C}_{31}$ | 100 |
| :---: | :---: |
| $\mathrm{C}_{32}$ | 105 |
| $\mathrm{C}_{33}$ | 30 |
| $\mathrm{C}_{1}^{\mathrm{m}}$ | 0.39 |
| $\mathrm{C}_{1}^{\mathrm{pw1}}$ | 0.45 |
| $\mathrm{C}_{1}^{\mathrm{pw} 2}$ | 0.82 |
| $\mathrm{C}_{1}^{\mathrm{pw} 3}$ | 0.75 |
| $\delta$ | 0.7 |
| $\mathrm{v}_{11}$ | 1.581 |


| $\mathrm{C}_{2}^{\mathrm{pw} 1}$ | 0.58 | $\mathrm{C}_{\mathrm{ml}}^{\text {sw1 }}$ | 0.500 |
| :---: | :---: | :---: | :---: |
| $\mathrm{C}_{2}^{\text {pw2 }}$ | 0.26 | $\mathrm{C}_{\mathrm{sw} 1}^{\mathrm{pw} 1}$ | 0.018 |
| $\mathrm{C}_{2}^{\text {pw3 }}$ | 0.53 | $\mathrm{C}_{\mathrm{m} 2}^{\text {sw } 2}$ | 0.160 |
| $\mathrm{C}_{\text {tm1 }}$ | 18 | $\mathrm{C}_{\mathrm{sw} 2}^{\mathrm{pw} 2}$ | 0.018 |
| $\mathrm{C}_{\mathrm{tm} 2}$ | 20 | $\mathrm{C}_{\mathrm{m} 3}^{\text {pw3 }}$ | 0.100 |
| $\mathrm{C}_{\mathrm{tm} 3}$ | 16 | FC | 0.06 |
| $\mathrm{C}_{\mathrm{tml}}^{\prime}$ | 0.7 | $\mathrm{S}_{1}$ | 175 |
| $\mathrm{C}_{\mathrm{tm} 2}^{\prime}$ | 0.21 | $\mathrm{S}_{2}$ | 184 |
| $\mathrm{V}_{22}$ | 1.342 | $\mathrm{V}_{33}$ | 1.49 |

Table-I1 (Optimal results for case-1)

|  | Scenario-1 | Scenario-II |  | Scenario-I | Scenario-II |
| :---: | :---: | :---: | :---: | :---: | :---: |
| PROF | 2179.272 | 2256.619 | $\mathrm{t}_{2}$ | 0.022 | 0.022 |
| $\mathrm{S}_{\mathrm{w}}^{1}$ | 339.694 | 341.077 | T | 4.943 | 3.733 |
| $\mathrm{S}_{\mathrm{w}}^{2}$ | 922.000 | 784.000 | $\mathrm{t}_{11}$ | 0.944 | 0.958 |
| $\mathrm{S}_{\mathrm{w}}^{3}$ | 1013.313 | 726.640 | $\mathrm{t}_{12}$ | 2.802 | 2.816 |
| $\mathrm{K}_{1}$ | 17.781 | 18.057 | $\mathrm{t}_{13}$ | 3.719 | 3.733 |
| $\mathrm{K}_{2}$ | 46.000 | 36.800 | $\mathrm{t}_{21}$ | 3.286 | 2.628 |
| $\mathrm{K}_{2}^{\prime}$ | 4.600 | 4.600 | $\mathrm{t}_{22}$ | 4.162 | 3.505 |
| $\mathrm{n}_{1}$ | 5 | 5 | $\mathrm{t}_{23}$ | 4.390 | 3.733 |
| $\mathrm{n}_{2}$ | 15 | 15 | $\mathrm{t}_{32}$ | 3.918 | 2.733 |
| $\mathrm{m}_{2}$ | 10 | 8 | $\mathrm{t}_{33}$ | 4.943 | 3.733 |
| $\mathrm{m}_{3}$ | 40 | 40 | $\mathrm{R}_{1}$ | 88.263 | 0 |
| $\mathrm{t}_{1}$ | 0.189 | 0.192 | $\mathrm{R}_{2}$ | 0.387 | 0 |
| $\mathrm{t}_{1}$ | 0.219 | 0.175 | $\mathrm{R}_{3}$ | 0 | 0 |

Following (33) to (36), problem (30) for case-2 is solved using a gradient-based non-linear optimization technique and the results are presented in the following tables:

Table-III (Individual minimum and maximum of objective functions)

| Objective <br> functions | Minimum |  | Maximum |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Scenario-I | Scenario-II |  | Scenario-I | Scenario-II |
| $\mathrm{PF}_{1}$ | 296.415 | 410.742 |  | 432.257 | 429.068 |
| $\mathrm{PF}_{2}$ | 54.315 | 561.979 |  | 592.954 | 592.964 |
| $\mathrm{PF}_{3}$ | 543.718 | 1296.017 |  | 1315.530 | 1326.930 |

Table-IV (Input Data for $\mathrm{PF}_{\mathrm{K}}^{1}, \mathrm{PF}_{\mathrm{K}}^{0}$ )

| Scenario | $\mathrm{PF}_{1}^{0}$ | $\mathrm{PF}_{1}^{1}$ | $\mathrm{PF}_{2}^{0}$ | $\mathrm{PF}_{2}^{1}$ | $\mathrm{PF}_{3}^{0}$ | $\mathrm{PF}_{3}^{1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| I | 296.416 | 432.057 | 54.316 | 592.954 | 543.719 | 1315.530 |
| II | 410.743 | 429.068 | 561.980 | 592.964 | 1296.018 | 1326.930 |

Table-V (Input Data for $\bar{\mu}_{\mathrm{PF}_{1}}, \bar{\mu}_{\mathrm{PF}_{2}}, \bar{\mu}_{\mathrm{PF}_{3}}$ )

| Scenario | $\bar{\mu}_{\mathrm{PF}_{1}}$ | $\bar{\mu}_{\mathrm{PF}_{2}}$ | $\bar{\mu}_{\mathrm{PF}_{3}}$ |
| :---: | :---: | :---: | :---: |
| I | 0.94 | 0.96 | 0.89 |
| II | 0.94 | 0.96 | 0.89 |

Let, with the above values (table-IV and -V), the membership functions of the objective functions may be formed of the types as per the Table-VI.

## Table-VI (Possible types of MF for objective functions)

| Objective functions | Type of membership functions |
| :---: | :---: |
| $\mathrm{PF}_{1}$ | Type-I or Type-II or Type-III |
| $\mathrm{PF}_{2}$ | Type-I or Type-II or Type-III |
| $\mathrm{PF}_{3}$ | Type-I or Type-II or Type-III |

Let, at the beginning, analysis is performed to find optimum $\lambda$ with the membership functions $\mathrm{PF}_{1}, \mathrm{PF}_{3}$ as linear (Type-II) and $\mathrm{PF}_{2}$ as Quadratic (Type-I). The optimum value of $\lambda$ is presented in Table-VII.

Table-VII (Optimal value of $\lambda$ )

| Minimum $\lambda$ | Scenario-I | Scenario-II |
| :---: | :---: | :---: |
| $\lambda^{*}$ | 0.23073736 | 0.4280142 |

With this value of $\lambda *$, the objective function $\mathrm{PF}_{3}$ is chosen as the most significant one and optimized The optimum results are:

Table-VIII (Optimal results when $\mathrm{PF}_{3}$ is chosen as the most important objective function )

|  | Scenario-1 | Scenario-II |  | Scenario-I | Scenario-II |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{PF}_{1}$ | 428.302 | 429.032 | $\mathrm{t}_{11}$ | 1.201 | 0.672 |
| $\mathrm{PF}_{2}$ | 539.499 | 575.633 | $\mathrm{t}_{12}$ | 3.141 | 2.612 |
| $\mathrm{PF}_{3}$ | 1313.993 | 1324.519 | $\mathrm{t}_{1}$ | 0.401 | 0.134 |
| T | 4.112 | 3.569 | $\mathrm{t}_{1}$ | 0.219 | 0.164 |
| $S_{\text {w }}^{1}$ | 359.198 | 311.456 | $\mathrm{t}_{2}$ | 0.0219 | 0.018 |
| $S_{\text {w }}^{2}$ | 829.999 | 749.500 | $\mathrm{t}_{13}$ | 4.098 | 3.569 |
| $\mathrm{S}_{\mathrm{w}}^{3}$ | 912.967 | 771.489 | $\mathrm{t}_{21}$ | 2.848 | 2.464 |
| $\mathrm{K}_{1}$ | 36.136 | 12.133 | $\mathrm{t}_{22}$ | 3.724 | 3.340 |
| $\mathrm{K}_{2}$ | 45.999 | 34.500 | $\mathrm{t}_{23}$ | 3.952 | 3.569 |
| $\mathrm{K}_{2}^{\prime}$ | 4.6 | 3.833 | $\mathrm{t}_{32}$ | 3.179 | 2.636 |
| $\mathrm{n}_{1}$ | 3 | 5 | $\mathrm{t}_{33}$ | 4.112 | 3.569 |
| $\mathrm{n}_{2}$ | 13 | 15 | $\mathrm{R}_{1}$ | 0.983 | 0 |
| $\mathrm{m}_{2}$ | 10 | 9 | $\mathrm{R}_{2}$ | 0.112 | 0 |
| $\mathrm{m}_{3}$ | 40 | 40 | $\mathrm{R}_{3}$ | 0.001 | 0 |

Now, the results obtained from Table-VIII are tested for Pareto-optimality according as (30) and are strong Pareto-optimum and hence can be accepted (Pareto-optimality results are not shown).
Still, if the decision-maker / practitioner is not satisfied with the outputs, he / she may perform the above analysis again by changing the current reference membership values for each of the membership function (i.e., $\bar{\mu}_{\mathrm{F}_{\mathrm{L}}}, \bar{\mu}_{\mathrm{F}_{\mathrm{C}}}, \bar{\mu}_{\mathrm{F}_{\mathrm{R}}}$ ) or /and choosing the membership function for $\mathrm{PF}_{1}, \mathrm{PF}_{2}$ and $\mathrm{PF}_{3}$. If this second time analysis does not also give the desired result, DM perform the analysis with the other possible different combination (in this case $3^{3}$ times) of the membership function with the same reference membership values for each of the membership function and this process is continued till the DM is satisfied.

## 8. Conclusion

In India, one particular brand of computers are brought by a multi-national from Singapore and stored in Pandicheri, a place in Southern India where a lot of financial concessions are available. From this depot, materials are transported to different cities for sale as per requirement/ demand. Such a reallife inventory distribution is portrayed here and mathematically modeled. It is solved using a gradientbased optimization technique and optimum results are presented. For the inventory practitioners, a manmachine interaction technique is demonstrated and assuming a DM's choice/preference, an optimum result is presented for case-2. Hence, the present formulation and analysis will be use full to multinational companies for their business in developing countries. The formulation of the model and analysis are quit general. These can be extended to include other features of inventory model such as deterioration, discount, inflation etc.

## Reference

1. Gupta, R. and Vrat(1986), P., An EOQ model for stock dependent consumption rate, Opsearch, 23, 1924.
2. Mandal, B. N. and Phaujder, S. (1989), An inventory model for deteriorating items and stock dependent consumption rate, Journal of Oprational Research Society, 40, 483-488.
3. Mandal, B. N. and Phaujder, S. (1989), A note on an inventory model with stock dependent consumption rate, Opsearch, 26, 43-46.
4. Urban, T. L. (1992), Deterministic inventory models incorporating marketing decisions, Computers and Engineering 22, 85-93.
5. Urban, T. L. (1992), inventory models with demand rate dependent on stock and shortage levels, International Journal of Production of Economics, 40, 21-28.
6. Bhunia, A. K. and Maiti, M. (1995),A deterministic two storage inventory model for variable production and inventory level dependent demand rate, Cahiers du CERO,37, 17-24.
7. Goyal, S. K., Gunasekaran, A (1995), An integrated production inventory marketing model for deteriorating item, Computers and Industrial Engeneering, 28,755-762.
8. Abad, P.L.(1996): Optimal pricing and lot-sizing under conditions perishability and partial backordering, Management Science, 42, 1093-1104.
9. R. V. Hartely (1976), Operations research-a managerial emphasis, Santa Monica, CA:Goodyear Publishing Company, 315-317, [chapter 12].
10. K. V. S. Sarma (1983), a deterministic inventory model with two levels of storage and an optimum release rule, Opsearch, 29, 175-180.
11. A. Goswami, K. S. Chaudhuri (1992), An economic order quantity model for items with two levels of storage for a linear trend in demand, J. Opl. Res. Soc., 43, 157-167.
12. T.P.M Pakkala and K.K Achary (1992a), A deterministic inventory model for deteriorating items with two warehouses and finite replenishment rate, European Journal of Operational Research, 57, 71-76.
13. T.P.M Pakkala and K.K Achary (1992b), Discrete time inventory model for deteriorating items with two warehouses, Opsearch, 29, 90-103.
14. A. K. Bhunia and M. Maiti (1994), A two warehouses inventory model for a linear trend in demand , Opserarch, 31, 318-329.
15. A. K. Bhunia and M. Maiti (1998), A two warehouses inventory model for deteriorating items with a linear trend in demand and shortages, J. Opl. Res. Soc., 49, 287 - 292.
16. M. Sakawa, H. Yano and T. Yumine, An Interactive Fuzzy Satisficing Method for Multi-objective Linear Programming Problems and Its Application, IEEE Transactions on Systema, Man and Cybernatics, July/August 1987, SMC-17, 654-661.
17. R.E. Bellman, and L.A. Zadeh, Decision-making in a fuzzy environment, Management Science, 17(1970), B141-B164.
18. H.Z. Zimmermann, Fuzzy linear programming with several objective functions, Fuzzy Sets and Systems, 1(1978), 46-55.
