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An Error Analysis of Polynomial Form Dead Reckoning Model based on a Numerical Analysis

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Abstract: This paper aims to analyze an error of dead reckoned data generated from received data in discrete temporal axis. That is, comparing with the data received temporally continuously, data acquired in every temporal interval has a certain degradation or uncertainty of information. Our purpose is to introduce the mathematical model of this degradation with regard to the metrics of temporal interval. Polynomial models are introduced for dead reckoning of the data between frames using a parameter calculated from the data over the last several frames. By employing the method of error analysis of a *numerical analysis* to the above polynomial model, theoretical models, which approximate the degree of degradation in accuracy based on parameters such as update interval and change of the data, are found. This paper also confirms the adaptability of the theoretical model by conducting simulation experiments generated by pen motion of writing string of letters by human. Experimental results show that the proposed theoretical model approximates well the average error in the simulation.

AMS Mathemarics Subject Classification: 68Q85, 68M07, 68Q25

Keywords: dead reckoning, update interval, numerical analysis, polynomial model.

1 Introduction

In recently, much research has come to be focused on the real-time system, and on their applications. In the real-time system, it is well known that the discrete-time property of information update due to the limitation in transmission capacity of the data effect to the interactivity, operability, and performance of the system [Ware,1994] [Ellis,1999] [Ellis,2002] [Hikichi,2002] [Singhal,1999]. The prediction method called dead reckoning is one of the method to solve the above problems, and has been applied in various type of systems.

This paper aims to analyze an error of extrapolated data generated from received data in discrete temporal axis. That is, comparing with the data received temporally continuously, discrete data acquired in every temporal interval has degradation or uncertainty of information. Our analysis is to introduce the mathematical model of this degradation with regard to the metrics of temporal interval.

A number of studies on the effects of using dead reckoning method in real-time systems have previously been reported. For example, Singhal et.al proposed an effective dead reckoning method in collaborative virtual environments [Singhal,1995] [Singhal,1996]. Ohlenburg proposed a method of collision detection between two objects on virtual space based on dead reckoning [Ohlenburg,2004]. Hikichi et.al applied a dead reckoning method in tactile communications systems, and examined the effects of operability[Hikichi,2002]. AMaze[Berglund,1985], DIVE[Hagsand,1996], and NPSnet[Macedonia,1994] introduced dead reckoning for the increase of real-time interactivity or reduction of the data transmission. Gutwin et.al investigated the effects of dead reckoning against temporal jitter of telepointer in real-time distributed groupware [Gutwin,2003]. Capin et.al applied a dead reckoning method to streaming virtual human animation on networked virtual environment [Capin,1997] [Capin,1999]. As explained in these reports, an utilization of dead reckoning are effective for the task performance in real-time system. However, in these reports, theoretical evaluation on the dead reckoning method has not been sufficiently done. In references [Hanawa,2002] [Hanawa,2004], we made theoretical studies of performance degradation and prediction method in collaborative tasks within a virtual environment. We did not, however, examine a theoretical analysis on a dead reckoning error. We therefore attempt, in this paper, to provide a formalization of the statistical error. We also compare our theoretical model with the results of simulation experiments and evaluate the validity of the model.

This paper is organized as follows. Section 2 explains a concept and mechanism of dead reckoning. In section 3, Lagrange polynomial model is introduced for dead reckoning of the data between frames using a parameter calculated from the data over the last several frames, and a theoretical models are proposed for approximating the dead reckoning error based on parameters of motion data and update interval. Finally, section 4 describes simulation experiments and shows that the proposed theoretical models agree well with the results of those simulations.

2 Model of discrete data with dead reckoning

Suppose that $A(t)$ is a function, whose parameter is a continuous time variable t. $A(t)$ is assumed to be a continuous and differentiable function of variable t in its domain. Specifically, in the real-time system, physical data such as the spatial position of manipulator, and the position of virtual user in virtual space, are presumed as $A(t)$. Suppose that the update interval of $A(t)$ is limited by the limitation of bandwidth, mechanical sampling rate, etc, only function value $A(t_i)$ in the discrete time $t_i (i = 0, 1, 2, 3...)$ can be observed as the update data in the user's side. In this case, if an update interval u between discrete time (the following update interval) can be assumed to be small enough, a change in $A(t)(t_i \leq t \leq t_{i+1})$ between two frames which continued on a temporal axis can be approximated by the latest value $A(t_i)$.

On the contrary, if u can not be assumed sufficiently small, smoothness of a change in $A(t)$ is lost in proportion to the update interval. A number of reports showed that increase in update interval degrades task performance in remote operations and remote collaborative tasks [Ware,1994] [Ellis,1999] [Ellis,2002] [Hikichi,2002] [Singhal,1999]. Dead reckoning is a method for the data extrapolation between update intervals by using prediction in data receiving side. Figure 1 shows a model of the observation for the temporally discrete data with the dead reckoning. In this figure, circle marks show the update data. Solid lines and dotted lines show path of true data and that of dead reckoned data respectively. Dashed areas show cumulative error of dead reckoning between two frames. As shown in Figure 1, the system extrapolates data between update interval by more detailed than a frame rate, and data can be presented in the detailed time interval. Today, dead reckoning is employed in many real-time systems [Singhal,1999]. Especially, in the field of networked virtual environment, the standardization of the communication protocol for dead reckoning has been reported in [IEEE1278,1993].

On the other hand, various type of interpolation polynomials are often used for the prediction model of dead reckoning, and it is known that even the low order model can get some effect in the experimental knowledge [Singhal,1995] [Singhal,1996] [Gutwin,2003] [Hikichi,2002] [Ohlenburg,2004]

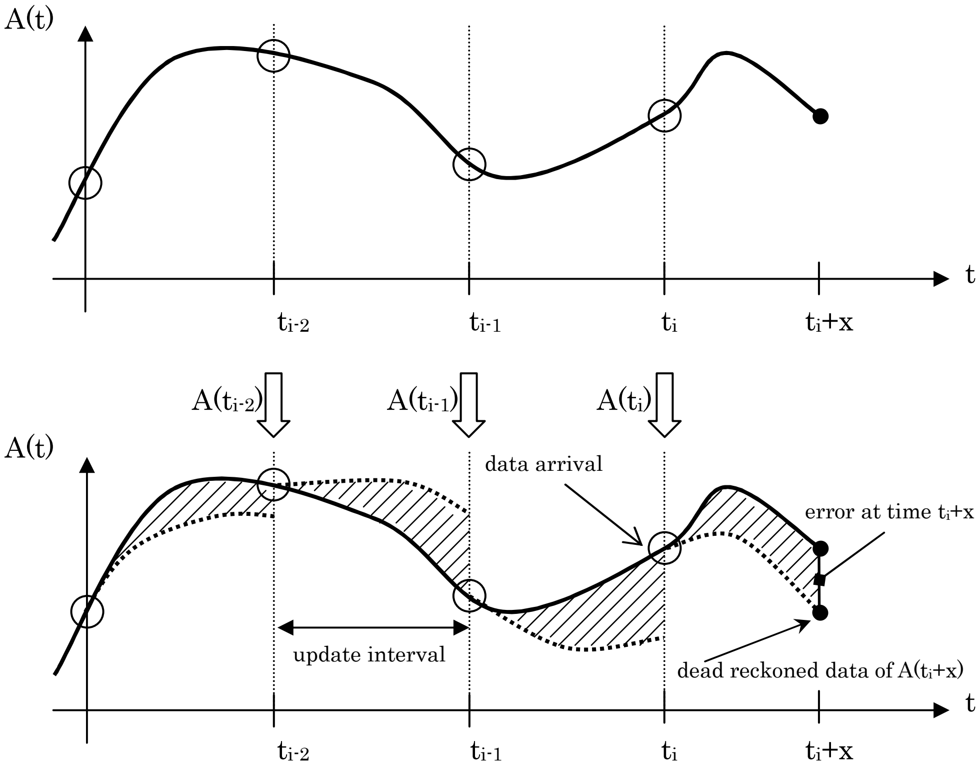

Figure 1: A model of the observation and the generated error with the dead reckoning

[Durbach,1998]. Furthermore, in the field of numerical analysis, some theoretical properties on the error of interpolation have been shown[Neumaier,2001] [Pozrikidis,1998]. However, theoretical analysis on the property of dead reckoning method based on interpolation polynomial model have not been sufficiently done. Therefore, it was difficult to estimate the most suitable update interval in beforehand in the systems utilizing dead reckoning. In the next section, a basic interpolation polynomial model is employed as the model of dead reckoning, and a method of error analysis of a numerical analysis is employed for the analysis of the error based on this model.

3 Error analysis of dead reckoning

3.1 Dead reckoning by polynomial models

Now, as a model of common dead reckoning method, we introduce Lagrange polynomial model [Neumaier,2001] [Pozrikidis,1998] that use the several frames of received data as parameters.

$$L_u(A(t), t_i)_n = \begin{cases} A(t_i) & (n = 0) \\ \sum_{j=0}^{n} \left(A(t_{i-j}) \prod_{k=0(k\neq j)}^{n} \frac{t - t_{i-k}}{t_{i-j} - t_{i-k}} \right) & (n \geq 1) \end{cases} \tag{1}$$

Here, t, t_i, u, and n denote the time($t \geq t_i$), the time of the latest data, update interval and order of the polynomial, respectively.

In this paper, we consider polynomial models of order 0, 1, and 2 (a polynomial model of order m is denoted below as an "mth-order-model"). Furthermore, for simplicity, we consider A(t) with a dimension of one. By applying the following discussion to each element of the data, this can easily be extended to two or more dimensions data. These models are described as following.

$$L_u(A(t_i + x), t_i)_0 = A(t_i) \qquad (n = 0) \tag{2}$$

$$L_u(A(t_i + x), t_i)_1 = A(t_i) + \frac{A(t_i) - A(t_{i-1})}{u} x \qquad (n = 1) \tag{3}$$

$$\begin{aligned} &L_u(A(t_i + x), t_i)_2 \\ &= A(t_{i-1}) + \frac{A(t_i) - A(t_{i-2})}{2u}(x + u) \\ &+ \frac{A(t_i) - 2A(t_{i-1}) + A(t_{i-2})}{2u^2}(x + u)^2 \qquad (n = 2) \end{aligned} \tag{4}$$

From the above models, dead reckoning method based on the $0th$-order model assumes that the value at time $t_i + x (0 \leq x \leq u)$ is the same as the latest value $A(t_i)$. Next, dead reckoning method based on $1st$-order model assumes that the value at time $t_i + x$ changes as a result of uniform motion. It changes, in other words, while maintaining the velocity observed over the last two frames expressed as

$$A_u^{(1)}(t_i) = \frac{A(t_i) - A(t_{i-1})}{u} \tag{5}$$

Moreover, dead reckoning method based on $2nd$-order model assumes that the value at time $t_i + x$ changes as a result of fixed acceleration motion. It changes, in other words, while maintaining the acceleration observed over the last three frames expressed as

$$A_u^{(2)}(t_{i-1}) = \frac{A(t_i) - 2A(t_{i-1}) + A(t_{i-2})}{u^2} \tag{6}$$

3.2 Approximation models for dead reckoning error

We attempt to analyze the dead reckoning error in previous subsection by the method of error analysis on numerical differentiation [Neumaier,2001] [Pozrikidis,1998]. In the following discussion, N denotes the total number of elapsed frames in observation.

The cumulative error from the time t_i to t_{i+1} generated between value $L_u(A(t_i + x), t_i)_n$ in the time $t_i + x$ predicted by the nth-order model and value $A(t_i + x)$ is equivalent to the cumulative of $|A(t_i + x) - L_u(A(t_i + x), t_i)_n|$ generated between frames. Accordingly, denoting the average of the cumulative error per unit time as $error(A(t), u, n)$, we have

$$error(A(t), u, n) = \frac{1}{Nu} \sum_{i=0}^{N-1} \int_0^u |A(t_i + x) - L_u(A(t_i + x), t_i)_n| dx \tag{7}$$

Since $A(t)$ can be assumed to be a continuous and differentiable function of a variable t, we can consider the Taylor expansion of $A(t)$ around $t = t_i$. The magnitude of that error $|A(t_i+x) - L_u(A(t_i+x), t_i)_n|$ by $0th$-order model can therefore be calculated as follow.

$$|A(t_i + x) - L_u(A(t_i + x), t_i)_0| = \left| \frac{A^{(1)}(t_i)}{1!} x + \frac{A^{(2)}(t_i)}{2!} x^2 + \dots \right| \tag{8}$$

By assuming that the second term of the right side above can be neglected, the cumulative error between the time t_i and t_{i+1} can be replaced by the update interval u and $A^{(1)}(t_i)$.

$$\int_0^u |A(t_i + x) - L_u(A(t_i + x), t_i)_0| \, dx \approx \frac{|A^{(1)}(t_i)|}{2} u^2 \tag{9}$$

From this, $error(A(t), u, 0)$ can be expressed as follows denoting the average of $|A^{(1)}(t_i)|$ as $\overline{|A^{(1)}|}$.

$$error(A(t), u, 0) = \frac{\overline{|A^{(1)}|}}{2} u \tag{10}$$

Now, the error generated as a result of dead reckoning using $1st$-order model is equivalent to divergence from uniform motion, i.e., to acceleration or higher temporal differential terms that occurs between frames. By using Taylor expansion of $A(t_{i-1})$ around $t = t_i$, we have

$$A(t_{i-1}) = A(t_i) - A^{(1)}(t_i)u + \frac{1}{2!}A^{(2)}(t_i)u^2 - \frac{1}{3!}A^{(3)}(t_i)u^3 + \dots \tag{11}$$

From this, eq.(5) can be expressed as follow [Neumaier,2001].

$$A_u^{(1)}(t_i) = A^{(1)}(t_i) - \frac{1}{2!}A^{(2)}(t_i)u + \frac{1}{3!}A^{(3)}(t_i)u^2 + \dots \tag{12}$$

The magnitude of that error $|A(t_i + x) - L_u(A(t_i + x), t_i)_1|$ by $1st$-order model can therefore be calculated as follows.

$$\begin{aligned} &|A(t_i + x) - L_u(A(t_i + x), t_i)_1| \\ &= \left| \frac{A^{(2)}(t_i)}{2!} x(x + u) + \frac{A^{(3)}(t_i)}{3!} x^2(x - u) + \dots \right| \end{aligned} \tag{13}$$

By the same way in eq.(9), the cumulative error between the time t_i and t_{i+1} can be replaced by the update interval u and $A^{(2)}(t_i)$.

$$\int_0^u |A(t_i + x) - L_u(A(t_i + x), t_i)_1| \, dx \approx \frac{5|A^{(2)}(t_i)|}{12} u^3 \tag{14}$$

From this, $error(A(t), u, 1)$ can be expressed as follows denoting the average of $|A^{(2)}(t_i)|$ as $\overline{|A^{(2)}|}$.

$$error(A(t), u, 1) = \frac{5\overline{|A^{(2)}|}}{12} u^2 \tag{15}$$

Moreover, the error generated as a result of dead reckoning using $2nd$-order model is equivalent to divergence from stable acceleration motion, i.e., to jerk or higher temporal differential terms that occurs between frames. By using Taylor expansion of $A(t_{i-2})$ around $t = t_{i-1}$ and that of $A(t_i)$ around $t = t_{i-1}$, we have

$$A(t_{i-2}) = A(t_{i-1}) - A^{(1)}(t_{i-1})u + \frac{1}{2!}A^{(2)}(t_{i-1})u^2 - \frac{1}{3!}A^{(3)}(t_{i-1})u^3 + \dots \tag{16}$$

$$A(t_i) = A(t_{i-1}) + A^{(1)}(t_{i-1})u + \frac{1}{2!}A^{(2)}(t_{i-1})u^2 + \frac{1}{3!}A^{(3)}(t_{i-1})u^3 + \dots \tag{17}$$

From the result of adding both sides of eq.(16) and eq.(17) and dividing by u^2, eq.(6) can be expressed as follow [Neumaier,2001].

$$A_u^{(2)}(t_{i-1}) = A^{(2)}(t_{i-1}) + \frac{2}{4!}A^{(4)}(t_{i-1})u^2 + \frac{2}{6!}A^{(6)}(t_{i-1})u^4 + \ldots \tag{18}$$

Also from the result of subtracting eq.(16) from eq.(17) and dividing by $2u$, we have

$$\frac{A(t_i) - A(t_{i-2})}{2u} = A^{(1)}(t_{i-1}) + \frac{1}{3!}A^{(3)}(t_{i-1})u^2 + \frac{1}{5!}A^{(5)}(t_{i-1})u^4 + \ldots \tag{19}$$

From the Taylor expansion of $A(t_i + x)$ around $t = t_{i-1}$, the magnitude of that error $|A(t_i+x) - L_u(A(t_i+x), t_i)_2|$ by $2nd$-order model can therefore be calculated as follows.

$$\begin{aligned} &|A(t_i + x) - L_u(A(t_i + x), t_i)_2| \\ &= \left| \frac{A^{(3)}(t_{i-1})}{3!}x(x+u)(x+2u) + \frac{A^{(4)}(t_{i-1})}{4!}x(x+u)^2(x+2u) + \ldots \right| \end{aligned} \tag{20}$$

By the same way in eq.(9), the cumulative error between the time t_i and t_{i+1} can be replaced by the update interval u and $A^{(3)}(t_{i-1})$.

$$\int_0^u |A(t_i + x) - L_u(A(t_i + x), t_i)_2| \, dx \approx \frac{3|A^{(3)}(t_{i-1})|}{8}u^3 \tag{21}$$

From this, $error(A(t), u, 2)$ can be expressed as follows denoting the average of $|A^{(3)}(t_{i-1})|$ as $\overline{|A^{(3)}|}$.

$$error(A(t), u, 2) = \frac{3\overline{|A^{(3)}|}}{8}u^3 \tag{22}$$

The following section describes simulation experiments to evaluate the workability of the approximation models expressed by Eq.(10), (15), and (22) above.

4 Simulation experiments

4.1 Design of experiments

Simulation experiments have been conducted to evaluate the workability of the theoretical models (eq.(10),(15),and (22)), and its adaptable range. In the experiments, the system calculates the error (the dashed area in Figure

1) based on eq.(7) from the generated pattern. This error is referred to as "average error" in units of *pixel*. Accuracy of the approximation of the dead reckoning error by theoretical models have been evaluated from this metrics.

Pen motion pattern [Kamada,2004] was employed as the model for motion data. The pen motion of writing string of letters by human was employed here as the most fundamental motion of the human hand. In the experiments, English words which consists of five letters to eight letters, were drawn by examinees on the screen, and a pen tablet was used for the examinees to draw them. The historical data which consist of X position of the cursor (referred to as A(t) below) and the Y position of it (referred to as A(t) below) were recorded with the time stamps. Window size was established to 1200x800 *pixel* and sampling interval of the pen tablet was established to 20 *msec*. Examinees were solicited to draw a word in the whole of the window slowly and carefully.

After the drawing, average error was calculated from the above data. In the following, the pattern of X position and that of Y position are referred to as X-axis pattern and Y-axis pattern respectively. Eight kinds of English words , i.e. "apple", "column", "common", "Colorado", "Green", "Nevada", "paper", and "Solomon", were employed in the experiments. To keep a contiguous fashion, these did not contain letters such as i and t.

4.2 Results

The experiments done for four examinees and total 32 patterns were measured. Simulations are performed for five type of update intervals (100, 200, ..., 500 *msec*). Figures 2, 3 and 4 present the results of these simulations. Figure 2 shows the average error of the dead reckoning by Lagrange polynomial model. Figure 3 shows the accuracy of the approximation by theoretical models. Figure 4 shows the example of dead reckoning path by $2nd$-order model ("apple", $u = 200$ *msec*). The accuracy of the approximation by theoretical models to the simulations was calculated from the following.

$$Accuracy(A(t)) = \frac{|(error(A(t), u, n) - Model)|}{error(A(t), u, n)} \tag{23}$$

The velocity, acceleration, and jerk in X-axis pattern, and those in Y-axis pattern were 104.0 $pixel/sec$, 305.5 $pixel/sec^2$, and 988.0 $pixel/sec^3$, and 135.1 $pixel/sec$, 399.2 $pixel/sec^2$, and 1278.0 $pixel/sec^3$, respectively.

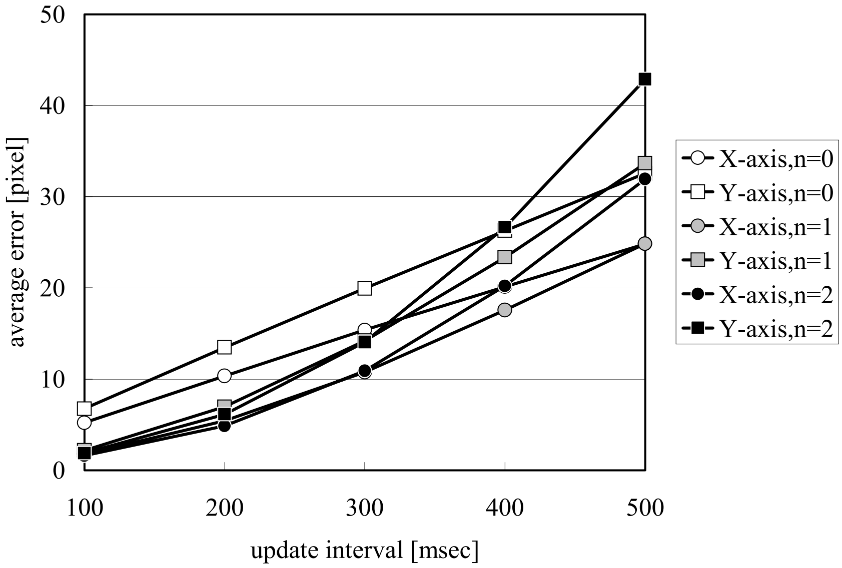

Figure 2: Comparison of the average error

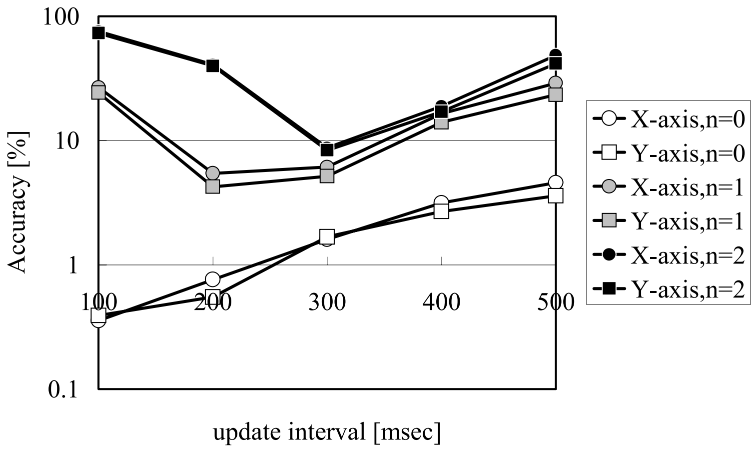

Figure 3: Comparison of the accuracy of approximation by theoretical model

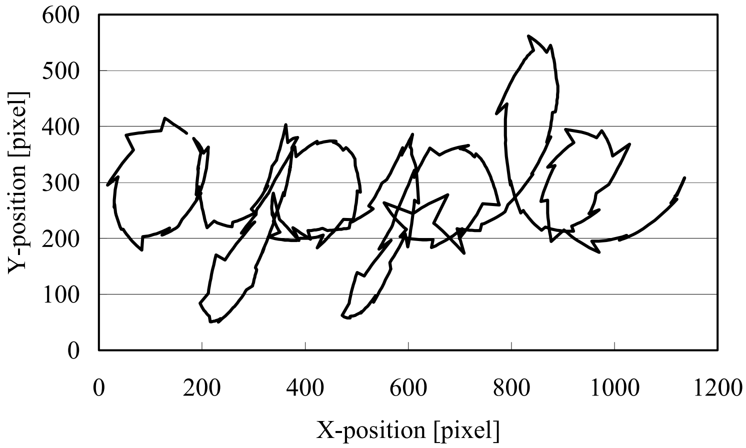

Figure 4: An example of the dead reckoned data (n=2,u=200 msec)

4.3 Discussion

The proposed theoretical models mostly approximated to the results of the simulations. Specifically, no major differences could be seen between the theoretical model and simulations.

In Figure 3, percentage discrepancy from the simulations was about 10%, for update interval 300 *msec* regardless of the order of polynomial. The percentage was over 10% in $2nd$-order model for update interval under 200 *msec* and $1st$-order model for update interval 100 *msec*. However, in Figure 2 and 4, the error for small update interval was found to be small. Therefore the discrepancy in such case is not severe in comparison with the case that the update interval is large. On the basis of these results, we can say that the dead reckoning error based on Lagrange polynomial model can be estimated by the proposed theoretical models.

Next, on comparing the results between the types of pattern, no major differences could be seen between the X-axis pattern and the Y-axis pattern (Figure 3). The reason for this is thought to be that the ratio of each of the average absolute value of derivative function were mostly same regardless of the type of data pattern.

On comparing the results between different order models, better results were obtained for the lower order model than the higher order model (Figure 3). The reason for this is thought to be that the contribution of higher order terms for error in higher order model is greater than that in lower order model since $1st$-order or $2nd$-order model contain the error generated from numerical differentiations.

All in all, the above results reveal that the proposed theoretical model approximates well simulations regardless of case.

5 Conclusion

This paper studied the relationship between dead reckoning error and update interval of the data in the real-time system based on numerical analysis. Specifically, the theoretical models were shown to be valid for estimating the polynomial type dead reckoning error.

First, the method of the dead reckoning based on the Lagrange polynomial model was employed, and relationship between the above error and update interval was studied based on the numerical analysis. As a result, theoretical models which showed the way that increase in the error caused by the increase in the update interval were shown.

Next, the validity of the proposed model was confirmed in the simulation experiment. Experimental results suggested that this model worked effectively to estimate the error of dead reckoning to finger movement and so on for the update interval under 300 *msec*.

The above findings demonstrate the usefulness of the models for estimating the dead reckoning error based on polynomial models.

In future work, we plan to make further studies on the applicability of the proposed theoretical model. For example, we plan to study an theoretical model based on a 3rd or higher order model and other form of dead reckoning method, and to examine the relation between the theoretical model and task performance.

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