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# AN EOQ MODEL FOR A DETERIORATING ITEM WITH TRENDED DEMAND, AND VARIABLE BACKLOGGING WITH SHORTAGES IN ALL CYCLES. 

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#### Abstract

The model describes an EOQ model with time-varying deterioration, partial backlogging which depends on the length of the waiting time for the next replenishment, linearly time-varying demand function over a finite time horizon and variable replenishment cycle. The model is solved analytically to obtain the optimal solution of the problem. It is illustrated with the help of a numerical example. Key Words: Inventory, economic order quantity, variable backlogging, trended demand, deterioration.


## 1. Introduction

In fomulating inventory models, the deterioration of items should not be neglected for all items. For examples, items like foodstuff, pharmaceuticals, chemicals, etc., deteriorate significantly. Many researchers like Aggarwal[1], Dave and Patel[2], Roychaudhuri and Chaudhuri[3], Dave[4], Bahari-Kashani[5], Wee[6], Hariga Al-Alyan[7], Benkherouf[8], etc., assumed that items deteriorates at constant rate.
The assumption of the constant deterioration rate was relaxed by Covert and Philip[9], who used a two-parameter Weibull distribution to represent the distribution of time to deterioration. This model was further generalized by Philip[10], by taking a three-parameter Weibull distribution. Misra[11] adopted a two-paremeter Weibull distribution deterioration to develop an inventory model with a finite rate of replenishment. These investigations were followed by several researchers like Deb and Chaudhuri[12], Goswami and Chaudhuri[13], Giri et al.[14], etc., where
the deterioration rate is considered to be time-proportional. The above researchers observed in several studies that the optimal condition does not depend on the parameters of the demand function under linearly increasing deterioration rate and varying service level. But Lin et al.[15] considered the problem of items with time-varying deterioration rate over a finite time-horizon, and found that the solutions are dependent on the form of demand function since the service level is assumed to be fixed herein. He assumed that the customers would wait for backlogging or a constant rate of customers would not wait. So during the shortage period, backlogging rate is a fixed fraction of the demand function. However the backlogging rate should depend on the time-spent in waiting for the arrival of the next lot. In the present paper, we assume time-varying deterioration, partial back-ordering that depends on the waiting time for backlogging and linear demand function over time horizon. An analytical solution of the model is discussed and it is illustrated with the help of a numerical example.

## 2. Assumptions and Notations

The following assumptions and notations are used to develop the proposed model:
(1) $H$ : Total time horizon for in the inventory system.
(2) $D(t)$ : The demand rate where $D(t)=a+b t, a>0, b \neq 0$.
(3) $s_{i}$ : The time at which shortage starts during the ith $(i=1,2, . ., n)$ cycle.
(4) $t_{i}$ : The time for the $i$ th replenishment, $i=1,2, . ., n$.
(5) $\theta(t)$ : Deterioration on the on-hand inventory per unit time, where $\theta(t)=\alpha \beta\left(t-t_{i}\right)^{\beta-1}, \alpha>0, \beta>0, t>t_{i}$.
(6) The rate of replenishment is infinite.
(7) Shortages are allowed except for the last cycle and each cycle starts with a shortage.
(8) $B(t)$ : Backlogging rate during shortage period which is dependent on the length of waiting time for the next replenishment and $B(t)=$ $\frac{1}{1+\delta\left(t_{i}-t\right)}, \delta \geq 0, t_{i}>t$.
(9) $I_{i 1}(t)$ : The inventory at any time $t \in\left[t_{i}, s_{i}\right], i=1,2, . ., n$.
(10) $I_{i 2}(t)$ : The shortage level at any time $t \in\left[s_{i-1}, t_{i}\right], i=1,2, . ., n$.
(11) $A$ : Ordering cost per replenishment cycle.
(12) $C_{h}$ : Inventory carrying cost per unit per unit time.
(13) $C_{s}$ : Shortage cost per unit per unit time.
(14) $C_{l}$ : Cost of lost in sales per unit.
(15) $C_{d}$ : Cost of deteriorated unit per unit.
(16) $n$ : Number of replenishments (shipment) for the buyer during the period H .
(17) $I_{i}$ : Total amount of inventory carried during the $i$ th $(i=1,2, . ., n)$ cycle.
(18) $D_{i}$ : Total number of deteriorated units during $i$ th $(i=1,2, . ., n)$ cycle.
(19) $S_{i}$ : Total shortage quantity during the $i$ th $(i=1,2, . ., n-1)$ cycle.
(20) $L_{i}$ : Total quantity of loss during the $i$ th $(i=1,2, . ., n-1)$ cycle.
(21) $T C$ : Total cost during the entire period $H$.

## 3. Formulation of the model and solution

The instantaneous inventory level $I_{i 1}(t)$ is given by the following differential equation

$$
\begin{equation*}
\frac{d I_{i 1}(t)}{d t}+\theta(t) I_{i 1}(t)=-D(t), \quad t_{i} \leq t \leq s_{i}, i=1,2, . ., n \tag{1}
\end{equation*}
$$

with the boundary condition $I_{i 1}\left(s_{i}\right)=0$.
Also for the period of shortages, the instantaneous shortage level $I_{i 2}$ is given by

$$
\begin{equation*}
\frac{d I_{i 2}(t)}{d t}=-D(t) B(t), \quad s_{i-1}<t \leq t_{i}, i=1,2, . ., n-1 \tag{2}
\end{equation*}
$$

with the initial condition $I_{i 2}\left(s_{i-1}\right)=0$.
The solution of (1) is given by

$$
\begin{equation*}
I_{i 1}(t)=e^{-\alpha\left(t-t_{i}\right)^{\beta}}\left\{\int_{t}^{s_{i}} e^{\alpha\left(u-t_{i}\right)^{\beta}}(a+b u) d u\right\} . \tag{3}
\end{equation*}
$$



Fig-1 :Pictorial representation of the inventory cycle .

Using Maclaurin series for $e^{x}$, (3) can be written as

$$
\begin{align*}
I_{i 1}(t) & =\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k} \alpha^{k+j}}{k!j!}\left[\frac{a+b t_{i}}{\beta j+1}\left(s_{i}-t_{i}\right)^{\beta j+1}\left(t-t_{i}\right)^{\beta k}\right. \\
& +\frac{b}{\beta j+2}\left(s_{i}-t_{i}\right)^{\beta j+2}\left(t-t_{i}\right)^{\beta k}-\frac{a+b t_{i}}{\beta j+1}\left(t-t_{i}\right)^{\beta k+\beta j+1} \\
& \left.-\frac{b}{\beta j+1}\left(t-t_{i}\right)^{\beta k+\beta j+2}\right] \tag{4}
\end{align*}
$$

where $t_{i} \leq t \leq s_{i}, \mathrm{i}=1,2, . ., \mathrm{n}$.
Therefore, the total amount of inventory carried during the interval $\left[t_{i}, s_{i}\right]$ is given by

$$
\begin{align*}
I_{i} & =\int_{t_{i}}^{s_{i}} I_{i 1}(t) d t \\
& =\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k} \alpha^{k+j}}{k!j!}\left[\frac{a+b t_{i}}{(\beta k+1)(\beta k+\beta j+2)}\left(s_{i}-t_{i}\right)^{\beta k+\beta j+2}\right. \\
& \left.+\frac{b}{(\beta k+1)(\beta k+\beta j+3)}\left(s_{i}-t_{i}\right)^{\beta k+\beta j+3}\right], i=1,2, \ldots, n . \tag{5}
\end{align*}
$$

The total number of units deteriorated during the interval $\left[t_{i}, s_{i}\right]$ is

$$
\begin{align*}
D_{i} & =I_{i 1}-\int_{t_{i}}^{s_{i}}(a+b t) d t \\
& =\sum_{j=0}^{\infty} \frac{\alpha^{j}}{j!}\left[\frac{a+b t_{i}}{\beta j+1}\left(s_{i}-t_{i}\right)^{\beta j+1}+\frac{b}{\beta j+2}\left(s_{i}-t_{i}\right)^{\beta j+2}\right] \\
& -a\left(s_{i}-t_{i}\right)-\frac{b}{2}\left(s_{i}{ }^{2}-t_{i}{ }^{2}\right), \quad i=1,2, \ldots, n . \tag{6}
\end{align*}
$$

The shortage level $I_{i 2}(t)$ for (2) is given by

$$
I_{i 2}(t)=\int_{s_{i-1}}^{t} D(u) B(u) d u
$$

$$
\begin{align*}
& =\int_{s_{i-1}}^{t} \frac{a+b u}{1+\delta\left(t_{i}-u\right)} d u \\
& =-\frac{1}{\delta^{2}}\left[\left(a \delta+b+b \delta t_{i}\right) \log \frac{\left(1+\delta\left(t_{i}-t\right)\right)}{\left(1+\delta\left(t-s_{i-1}\right)\right)}+b \delta\left(t-s_{i-1}\right)\right] \tag{7}
\end{align*}
$$

where $s_{i-1} \leq t \leq t_{i}, \quad i=1,2, ., n$.
The total quantity of shortage during the interval $\left[s_{i}, t_{i}\right]$ is

$$
\begin{align*}
S_{i} & =\int_{s_{i-1}}^{t_{i}} I_{i 2}(t) d t \\
& =\frac{a \delta+b+b \delta t_{i}}{\delta^{2}}\left\{\left(t_{i}-s_{i-1}\right)-\frac{1}{\delta} \log \left(1+\delta\left(t_{i}-s_{i-1}\right)\right)\right\} \\
& -\frac{b}{2 \delta}\left(t_{i}-s_{i-1}\right)^{2}, \quad i=1,2, \ldots, n \tag{8}
\end{align*}
$$

The total quantity of loss during the interval $\left[s_{i-1}, t_{i}\right]$ is

$$
\begin{align*}
L_{i} & =\int_{s_{i-1}}^{t_{i}}[D(t)-D(t) B(t)] d t \\
& =\int_{s_{i-1}}^{t_{i}}\left[(a+b t)-\frac{a+b t}{1+\delta\left(t_{i}-t\right)}\right] d t \\
& =a\left(t_{i}-s_{i-1}\right)+\frac{b}{2}\left(t_{i}{ }^{2}-s_{i-1}{ }^{2}\right)-\frac{a \delta+b+b \delta t_{i}}{\delta^{2}} \log \left(1+\delta\left(t_{i}-s_{i-1}\right)\right) \\
& +\frac{b}{\delta}\left(t_{i}-s_{i-1}\right) \tag{9}
\end{align*}
$$

Therefore the total variable cost over the time horizon H is given by

$$
\begin{aligned}
T C=n A & +C_{h} \sum_{i=1}^{n} I_{i}+C_{d} \sum_{i=1}^{n} D_{i}+C_{s} \sum_{i=1}^{n} S_{i}+C_{l} \sum_{i=1}^{n} L_{i} \\
=n A & +\sum_{i=1}^{n}\left\{C _ { h } \sum _ { k = 0 } ^ { \infty } \sum _ { j = 0 } ^ { \infty } \frac { ( - 1 ) ^ { k } \alpha ^ { k + j } } { k ! j ! } \left[\frac{a+b t_{i}}{(\beta k+1)(\beta k+\beta j+2)}\left(s_{i}-t_{i}\right)^{\beta k+\beta j+2}\right.\right. \\
& \left.+\frac{b}{(\beta k+1)(\beta k+\beta j+3)}\left(s_{i}-t_{i}\right)^{\beta k+\beta j+3}\right] \\
& +C_{d}\left\{\sum_{j=0}^{\infty} \frac{\alpha^{j}}{j!}\left[\frac{a+b t_{i}}{\beta j+1}\left(s_{i}-t_{i}\right)^{\beta j+1}+\frac{b}{\beta j+2}\left(s_{i}-t_{i}\right)^{\beta j+2}\right]\right. \\
& \left.-a\left(s_{i}-t_{i}\right)-\frac{b}{2}\left(s_{i}{ }^{2}-t_{i}^{2}\right)\right\} \\
& +C_{s}\left\{\frac{a \delta+b+b \delta t_{i}}{\delta^{2}}\left\{\left(t_{i}-s_{i-1}\right)-\frac{1}{\delta} \log \left(1+\delta\left(t_{i}-s_{i-1}\right)\right)\right\}\right. \\
& \left.-\frac{b}{2 \delta}\left(t_{i}-s_{i-1}\right)^{2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& +C_{l}\left\{a\left(t_{i}-s_{i-1}\right)+\frac{b}{2}\left(t_{i}{ }^{2}-s_{i-1}{ }^{2}\right)-\frac{a \delta+b+b \delta t_{i}}{\delta^{2}} \log \left(1+\delta\left(t_{i}-s_{i-1}\right)\right)\right. \\
& \left.\left.+\frac{b}{\delta}\left(t_{i}-s_{i-1}\right)\right\}\right\}
\end{aligned}
$$

Our aim is to determine $t_{i}$ and $s_{i}$ which would minimize the total variable cost $T C$ of the inventory system.
For a fixed n , the necessary conditions for $T C$ to be minimum are

$$
\begin{aligned}
& \frac{\partial T C}{\partial t_{i}}=0, \quad i=1,2, . ., n \\
& \frac{\partial T C}{\partial s_{i}}=0, \quad i=1,2, . ., n-1
\end{aligned}
$$

where

$$
\begin{align*}
\frac{\partial T C}{\partial t_{i}} & =C_{h} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k} \alpha^{k+j}}{k!j!}\left[\frac{b}{(\beta k+1)(\beta k+\beta j+2)}\left(s_{i}-t_{i}\right)^{\beta k+\beta j+2}\right. \\
& \left.-\frac{a+b s_{i}}{\beta k+1}\left(s_{i}-t_{i}\right)^{\beta k+\beta j+1}\right] \\
& +C_{d}\left\{\sum_{j=0}^{\infty} \frac{\alpha^{j}}{j!}\left[\frac{-b \beta j}{\beta j+1}\left(s_{i}-t_{i}\right)^{\beta j+1}-\left(a+b s_{i}\right)\left(s_{i}-t_{i}\right)^{\beta j}\right]+a+b t_{i}\right\} \\
& +C_{s}\left\{\frac{\left(a \delta+b+b \delta t_{i}\right)\left(t_{i}-s_{i-1}\right)}{\delta\left(1+\delta\left(t_{i}-s_{i-1}\right)\right)}-\frac{b}{\delta^{2}} \log \left(1+\delta\left(t_{i}-s_{i-1}\right)\right)\right\} \\
& +C_{l}\left\{a+b t_{i}-\frac{a \delta+b+b \delta t_{i}}{\delta\left(1+\delta\left(t_{i}-s_{i-1}\right)\right)}-\frac{b}{\delta} \log \left(1+\delta\left(t_{i}-s_{i-1}\right)\right)\right. \\
& \left.+\frac{b}{\delta}\right\} \tag{11}
\end{align*}
$$

and

$$
\begin{align*}
\frac{\partial T C}{\partial s_{i}} & =C_{h} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k} \alpha^{k+j}}{k!j!} \frac{a+b s_{i}}{\beta k+1}\left(s_{i}-t_{i}\right)^{\beta k+\beta j+1} \\
& +C_{d}\left\{\sum_{j=0}^{\infty} \frac{\alpha^{j}}{j!}\left(a+b s_{i}\right)\left(s_{i}-t_{i}\right)^{\beta j}-a-b s_{i}\right\} \\
& +C_{s}\left\{-\frac{\left(a \delta+b+b \delta t_{i+1}\right)\left(t_{i+1}-s_{i}\right)}{\delta\left(1+\delta\left(t_{i+1}-s_{i}\right)\right)}+\frac{b}{\delta}\left(t_{i+1}-s_{i}\right)\right\} \\
& +C_{l}\left\{-a-b s_{i}+\frac{\left(a \delta+b+b \delta t_{i+1}\right)}{\delta\left(1+\delta\left(t_{i+1}-s_{i}\right)\right)}-\frac{b}{\delta}\right\} . \tag{12}
\end{align*}
$$

For a fixed n, the sufficient condition for $T C$ to be minimum is that the Hessian matrix of $T C$ (i.e, $\nabla^{2} T C$ ) is positive definite where
where

$$
\begin{align*}
\frac{\partial^{2} T C}{\partial t_{i} \partial s_{i-1}}= & C_{s}\left\{-\frac{b \delta\left(t_{i}-s_{i-1}\right)+\left(a \delta+b+b \delta t_{i}\right)}{\delta\left(1+\delta\left(t_{i}-s_{i-1}\right)\right)}\right.  \tag{14}\\
& \left.+\frac{\left(a \delta+b+b \delta t_{i}\right)\left(t_{i}-s_{i-1}\right)}{\left(1+\delta\left(t_{i}-s_{i-1}\right)^{2}\right.}+\frac{b}{\delta}\right\} \\
& +C_{l}\left\{\frac{b}{\left(1+\delta\left(t_{i}-s_{i-1}\right)\right)}+\frac{a \delta+b+b \delta t_{i}}{\left(1+\delta\left(t_{i}-s_{i-1}\right)\right)^{2}}\right\}, \\
\frac{\partial^{2} T C}{\partial t_{i}{ }^{2}}= & C_{h} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k} \alpha^{k+j}}{k!j!}\left[\frac{a+b s_{i}}{\beta k+1}(\beta k+\beta j+1)\left(s_{i}-t_{i}\right)^{\beta k+\beta j}\right. \\
- & \left.\frac{b}{\beta k+1}\left(s_{i}-t_{i}\right)^{\beta k+\beta j+1}\right] \\
+ & C_{d}\left\{\sum_{j=0}^{\infty} \frac{\alpha^{j}}{j!}\left[\left(a+b s_{i} \beta j\left(s_{i}-t_{i}\right)^{\beta j-1}-b\left(s_{i}-t_{i}\right)^{\beta j}\right]+b\right\}\right. \\
+ & C_{s}\left\{\frac{\left(a \delta+b+b \delta t_{i}\right)+b \delta\left(t_{i}-s_{i-1}\right)}{\delta\left(1+\delta\left(t_{i}-s_{i-1}\right)\right)}\right. \\
- & \frac{\left(a \delta+b+b \delta t_{i}\right)\left(t_{i}-s_{i-1}\right)}{\left(1+\delta\left(t_{i}-s_{i-1}\right)\right)^{2}}-\frac{b\left(1+\delta\left(t_{i}-s_{i-1}\right)\right)}{\delta\left(1+b t_{i}\right.} \\
+ & C_{l}\left\{b-\frac{2 b}{1+\delta\left(t_{i}-s_{i-1}\right)}-\frac{a \delta+b+b \delta}{\left(1+\delta\left(t_{i}-s_{i-1}\right)\right)^{2}}\right\} \\
\frac{\partial^{2} T C}{\partial s_{i} \partial t_{i}}= & \frac{\partial^{2} T C}{\partial t_{i} \partial s_{i}}=C_{h} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k} \alpha^{k+j}}{k!j!}\left[-\frac{a+b s_{i}}{\beta k+1}(\beta k+\beta j+1)\left(s_{i}-t_{i}\right)^{\beta k+\beta j}\right] \\
& +C_{d} \sum_{j=0}^{\infty} \frac{\alpha^{j}}{j!}\left[-\left(a+b s_{i}\right) \beta j\left(s_{i}-t_{i}\right)^{\beta j-1}\right],
\end{align*}
$$

$$
\begin{aligned}
\frac{\partial^{2} T C}{\partial s_{i}{ }^{2}} & =C_{h} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k} \alpha^{k+j}}{k!j!}\left[\frac{b}{\beta k+1}\left(s_{i}-t_{i}\right)^{\beta k+\beta j+1}\right. \\
& \left.+\frac{a+b s_{i}}{\beta k+1}(\beta k+\beta j+1)\left(s_{i}-t_{i}\right)^{\beta k+\beta j}\right] \\
& +C_{d}\left\{\sum_{j=0}^{\infty}\left[\frac{(-1)^{j}}{j!} b\left(s_{i}-t_{i}\right)^{\beta} j+\left(a+b s_{i}\right) \beta j\left(s_{i}-t_{i}\right)^{\beta j-1}\right]-b\right\} \\
& +C_{s}\left\{\frac{\left(a \delta+b+b \delta t_{i+1}\right.}{\delta\left(1+\delta\left(t_{i+1}-s_{i}\right)\right)}+\frac{\left(a \delta+b+b \delta t_{i+1}\right)\left(t_{i+1}-s_{i}\right)}{\left(1+\delta\left(t_{i+1}-s_{i}\right)\right)^{2}}-\frac{b}{\delta}\right. \\
& +C_{l}\left\{-b+\frac{\left(a \delta+b+b \delta t_{i+1}\right)}{\left.1+\delta\left(t_{i+1}-s_{i}\right)\right)^{2}}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^{2} T C}{\partial s_{i} \partial t_{i+1}} & =C_{s}\left\{-\frac{\left(a \delta+b+b \delta t_{i+1}\right)}{\delta\left(1+\delta\left(t_{i+1}-s_{i}\right)\right)}\right. \\
& \left.+\frac{\left(a \delta+b+b \delta t_{i+1}\right)\left(t_{i+1}-s_{i}\right)}{\left(1+\delta\left(t_{i+1}-s_{i}\right)\right)^{2}}+\frac{b}{\delta\left(1+\delta\left(t_{i+1}-s_{i}\right)\right)}\right\} \\
& +C_{l}\left\{-\frac{\left(a \delta+b+b \delta t_{i+1}\right)}{\left(1+\delta\left(t_{i+1}-s_{i}\right)\right)^{2}}+\frac{b}{\left(1+\delta\left(t_{i+1}-s_{i}\right)\right.}\right\} .
\end{aligned}
$$

## 5. Solution Procedure

Before going to the solution procedure, we propose the following theorem.
Theorem: If $g\left(s_{i}, t_{i+1}\right)<f\left(s_{i}, t_{i}\right)<g\left(s_{i-1}, t_{i}\right), i=1,2, \ldots, n$, then $\nabla^{2} T C$ is positive definite, where

$$
\begin{aligned}
f(x, y) & =-C_{h} b\left(s_{i}-t_{i}\right) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k} \alpha^{k+j}}{k!j!} \frac{1}{\beta k+1}\left(s_{i}-t_{i}\right)^{\beta k+\beta j} \\
& -C_{d} b\left[e^{\alpha\left(s_{i}-t_{i}\right)^{\beta}}-1\right]
\end{aligned}
$$

and

$$
\begin{aligned}
g(x, y) & =C_{s}\left\{\frac{b(y-x)}{1+\delta(y-x)}\right\} \\
& +C_{l}\left\{-b+\frac{b}{1+\delta(y-x)}+\frac{2(a \delta+b+b \delta y)}{1+\delta(y-x)}\right\}
\end{aligned}
$$

Proof: The proof is given in the Appendix.
Using the above proposed theorem we find the solution as under.

1 Input the parameter values $H, a, b, \alpha, \beta, \delta, A, C_{h}, c_{s}, C_{l}, C_{d}$.
2 Initialize the number $t_{1}$. With this value of $t_{1}$ and $s_{0}(=0)$, find $s_{1}$ from (11).

3 Find $t_{2}$ from (12) using the values of $s_{1}$ and $t_{1}$.
4 Using $t_{2}$ and $s_{1}$, find $s_{2}$ from (11) and proceed in this way until $s_{n}$ is obtained.

5 If $s_{n}$ is very close to $H$ and if $t_{1}, s_{1}, t_{2}, s_{2}, \ldots, t_{n}, s_{n}$ satisfies $g\left(s_{i}, t_{i+1}\right)<$ $f\left(s_{i}, t_{i}\right)<g\left(s_{i-1}, t_{i}\right), i=1,2, \ldots, n$, then $t_{1}, s_{1}, t_{2}, s_{2}, \ldots, t_{n}, s_{n}$ is an optimal solution for n replenishment cycles and find $T C$.

6 If $s_{n}>H$, a smaller value of $t_{1}$ is initialized and if $s_{n}<H$, a larger value of $t_{1}$ is initialized.

7 For each value of $n=1,2,3, \ldots$, and so on, corresponding total costs are obtained to find the optimal solution.

## 5. Numerical Example:

Let $a=45, b=4, \alpha=0.2, \beta=0.6, \delta=4, H=1, A=250, C_{h}=40, C_{d}=200$, $C_{s}=80$ in appropriate units. To solve the nonlinear equations (11) and (12), we take help of the numerical computational software Mathematica (version 4.1). We obtain the optimal solutions $t_{i}$ and $s_{i}$ given in Table 1. It shows that number of replenishments is 4 and the corresponding minimum total cost is 66391.80 . The corresponding total costs for $n=1,2,3, \ldots$ are given in Table 2.
Table 1. Optimal solution of the numerical example

| $i$ | $t_{i}{ }^{*}$ | $s_{i}{ }^{*}$ |
| :---: | :---: | :---: |
| 1 | 0.488 | 0.78122 |
| 2 | 0.86569 | 0.94856 |
| 3 | 0.97269 | 0.99097 |
| 4 | 0.99804 | 1.00013 |

Table 2.Total cost for different replenishment cycles

| $i$ | $T C$ |
| :--- | :---: |
| 1 | 74074.40 |
| 2 | 67716.10 |
| 3 | 67591.90 |
| $4^{*}$ | $66391.80^{*}$ |
| 5 | 66474.20 |

## 6. Concluding remarks:

The present paper describes an inventory model where the main issues are to consider deterioration, backlogging and variable replenishment cycle. Although many researchers used constant or linear deterioration rate to simplify their models, it is now widely adopted and more realistic to take deterioration rate following Weibull distribution. A complete rate or a constant partial rate were used in many studies to describe the backlogging rate. But it is more realistic to assume the backlogging rate to be time proportional with waiting time of backlogging. Some inventory modellers used the backlogging rate to be an exponential function of waiting time. But in real practice, backlogging rate never varies so high as exponential. So, we have considered here the backlogging rate $\frac{1}{1+\delta\left(t_{i}-t\right)}$, which seems to be better. For the replenishment cycle, a fixed period is generally used in many studies. However, a variable replenishment cycle would be more efficient than a fixed replenishment cycle in terms of total costs.

## References

[1] Aggarwal, S. P., A note on an order-level inventory model for a system with constant rate of deterioration. Opsearch, 1979, 15, 184-187.
[2] Dave, U. and Patel, K., (T, $S_{i}$ ) policy inventory model for deteriorating items with time proportional demand. Journal of Operational Research Society, 1981, 32, 137-142.
[3] Roychowdhury, M. and Chaudhuri, K. S., An order-level inventory model for deteriorating items with finite rate of replenishment. Opsearch, 1983, 20, 99-106.
[4] Dave U., An order-level inventory model for deteriorating items with variable instantaneous demand and discrete opportunities for replenishment. Opsearch, 1986, 23(1), 244-249.
[5] Bahari-Kashani, H., Replenishment schedule for deteriorating items with time-proportional demand. Journal of Operational Research Society, 1989, 40, 75-81.
[6] Wee, H. M., A deterministic lot-size inventory model for deteriorating items with shortages and a declining market. Computers $\mathcal{\&}$ Operations Research, 1995, 22(3), 345-356.
[7] Hariga, M.A., and Al-Alyan, A., A lot sizing heuristic for deteriorating items with shortages in growing and declining markets. Computers and Operations Research,1997, 24(11), 1075-1083.
[8] Benkherouf, L., Note on a deterministic lot-size inventory model for deteriorating items with shortages and a declining market. Computers and Operations Research, 1998, 25(1), 63-65.
[9] Covert, R. P. and Philip, G. C., An EOQ model for items with Weibull distribution deterioration. AIIE Transactions, 1973, 5, 323-326.
[10] Philip, G. C., A generalized EOQ model for items with Weibull distribution deterioration. AIIE Transactions, 1974, 6, 159-162.
[11] Misra, R. B., Optimum production lot-size model for a system with deteriorating inventory. International Journal of Productions Research, 1975 13, 495-505.
[12] Deb, M., and Chaudhuri, K.S., An EOQ model for items with finite rate of production and variable rate of deterioration. Journal of the Operational Research Society, 1986, 23(1), 175-181.
[13] Goswami, A. and Chaudhuri, K. S., An EOQ model for deteriorating items with a linear trend in demand. Journal of Operational Research Society, 1991, 42(12), 1105-1110.
[14] Giri, B.C, Goswami, A., and Chaudhuri, K.S., An EOQ model for deteriorating items with time-varying demand and costs. Journal of the Operational Research Society, 1996, 47(11), 1398-1405.
[15] Lin, C., Tan, B. and Lee, W. C., An EOQ model for deteriorating items with time-varying demand and shortages. International Journal of System Science, 2000, 31(3), 39-400.
[16] Horn R., and Johnson C., Matrix Analysis . Cambridge University Press, 1990, 403.

## APPENDIX-A:

## Proof of the theorem:

Since $\left(s_{i-1}, t_{i}\right)$ and $\left(t_{i}, s_{i}\right)$ have negative correlation $\forall i=1,2,3, \ldots, n$, we have $\frac{\partial^{2} T C}{\partial t_{i} \partial s_{i-1}}<0, \frac{\partial^{2} T C}{\partial s_{i} \partial t_{i}}<0, \frac{\partial^{2} T C}{\partial t_{i+1} \partial s_{i}}<0, \forall i=1,2,3, \ldots, n$. Hence

$$
\frac{\partial^{2} T C}{\partial t_{i}{ }^{2}}-\left|\frac{\partial^{2} T C}{\partial s_{i-1} \partial t_{i}}\right|-\left|\frac{\partial^{2} T C}{\partial s_{i} \partial t_{i}}\right|
$$

$$
\begin{aligned}
= & -C_{h} b\left(s_{i}-t_{i}\right) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k} \alpha^{k+j}}{k!j!} \frac{1}{\beta k+1}\left(s_{i}-t_{i}\right)^{\beta k+\beta j} \\
& -C_{d} b\left[e^{\alpha\left(s_{i}-t_{i}\right)^{\beta}}-1\right] C_{s}\left\{\frac{b\left(t_{i}-s_{i-1}\right)}{1+\delta\left(t_{i}-s_{i-1}\right)}\right\} \\
& +C_{l}\left\{-b+\frac{b}{1+\delta\left(t_{i}-s_{i-1}\right)}+2 \frac{\left(a \delta+b+b \delta t_{i}\right)}{1+\delta\left(t_{i}-s_{i-1}\right)}\right\} \\
& =-f\left(s_{i}, t_{i}\right)+g\left(s_{i-1}, t_{i}\right)>0, \forall i=1,2,3, \ldots n
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\partial^{2} T C}{\partial s_{i}{ }^{2}}-\left|\frac{\partial^{2} T C}{\partial t_{i} \partial s_{i}}\right|-\left|\frac{\partial^{2} T C}{\partial t_{i+1} \partial s_{i}}\right| \\
= & C_{h} b\left(s_{i}-t_{i}\right) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k} \alpha^{k+j}}{k!j!} \frac{1}{\beta k+1}\left(s_{i}-t_{i}\right)^{\beta k+\beta j} \\
+ & C_{d} b\left[e^{\alpha\left(s_{i}-t_{i}\right)^{\beta}}-1\right]-C_{s}\left\{\frac{b\left(t_{i}-s_{i-1}\right)}{1+\delta\left(t_{i}-s_{i-1}\right)}\right\} \\
- & C_{l}\left\{-b+\frac{b}{1+\delta\left(t_{i}-s_{i-1}\right)}+2 \frac{\left(a \delta+b+b \delta t_{i}\right)}{1+\delta\left(t_{i}-s_{i-1}\right)}\right\} \\
= & f\left(s_{i}, t_{i}\right)-g\left(s_{i-1}, t_{i}\right)>0, \forall i=1,2,3, . . n .
\end{aligned}
$$

Therefore $\nabla^{2} T C$ is positive definite (Horn and Johnson[16]).

