

BOUNDS FOR THE MODIFIED ECCENTRIC CONNECTIVITY INDEX

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ABSTRACT. The modified eccentric connectivity index of a graph is defined as the sum of the products of eccentricity with the total degree of neighboring vertices, over all vertices of the graph. This is a generalization of eccentric connectivity index. In this paper, we derive some upper and lower bounds for the modified eccentric connectivity index in terms of some graph parameters such as number of vertices, number of edges, radius, minimum degree, maximum degree, total eccentricity, the first and second Zagreb indices, Weiner index etc.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph with n vertices and m edges. We denote the degree of a vertex v by $\deg(v)$ and the maximum and minimum degree of the graph G by Δ and δ respectively. The distance between the vertices u and v of G , is equal to the length, that is the number of edges of a shortest path connecting u and v and we denote it by $d(u, v)$. Let $D(v)$ be the sum of distances from v to all other vertices in G , i.e., $D(v) = \sum_{u \in V} d(v, u)$. For a given vertex v , its eccentricity $\varepsilon(v)$ is the largest distance from v to any other vertices of G . The radius and diameter of the graph are respectively the smallest and largest eccentricity among all the vertices of G , whereas, the total eccentricity of G , denoted by $\theta(G)$, is the sum of eccentricities of all the vertices of G . In recent years, different topological indices, based on degree and eccentricity of a vertex of a graph are subject to large number of studies. Among these, the eccentric connectivity index, proposed by Gupta et. al [10], defined as $\xi^c(G) = \sum_{v \in V} \deg(v)\varepsilon(v)$, is one of the most popular vertex degree and eccentricity based topological index and is subject to a large number of chemical as well as mathematical studies [1, 6, 22]. If $N(v) = \{v : uv = e \in E\}$, then the modified eccentric connectivity index of any graph is defined as

$$(1.1) \quad \xi_c(G) = \sum_{v \in V} \delta(v)\varepsilon(v)$$

where $\delta(v) = \sum_{u \in N(v)} \deg(u)$. It should be noted that $\delta(\cdot)$ is used in the context of a vertex, whereas, δ is a global parameter of the graph G .

Ashrafi et. al in [1] presented some graph operation of modified eccentric connectivity polynomial. However, no further study of this index is being recorded so far. To establish upper and lower bounds of different topological indices in terms of different graph invariants and parameters, there are various study of which only some recent results are mentioned here [1, 5, 6, 7, 8, 13, 14]. In this paper, first we find modified eccentric connectivity index of some particular graph, and then we investigate some new upper and lower bounds of modified eccentric connectivity index in terms of number of vertices (n), number of edges (m), maximum vertex degree (Δ), minimum vertex degree (δ), radius (r), diameter (d), total eccentricity ($\theta(G)$), the first Zagreb index ($M_1(G)$) [13], the second Zagreb index ($M_2(G)$) [2], the first Zagreb eccentricity index ($E_1(G)$) [20], Wiener Index ($W(G)$) [18, 19], Harary Index ($H(G)$) [3, 21] and eccentric connectivity index ($\xi^c(G)$) [22].

2. Main Results

From (1.1), it is clear that if all the vertices of G are of same eccentricity e , then $\xi_c(G) = eM_1(G)$. Similarly, if all the vertices of G are of same degree (k) and eccentricity (e) then $\xi_c(G) = nk^2e$. Using these results and from direct calculation, we can find the following explicit formulas for the eccentric connectivity index of different graphs.

Proposition 2.1. *Let $K_n, C_n, Q_m, \Pi_m, A_m$ denote the complete graph with n vertices, the cycle on n vertices, m -dimensional hypercube, m -sided prism and the m -sided antiprism respectively. Then the modified eccentric connectivity indices of these graphs are given as follows.*

$$\begin{aligned} (i) \quad & \xi_c(K_n) = n(n-1)^2. \\ (ii) \quad & \xi_c(C_n) = 4n \lfloor n/2 \rfloor. \\ (iii) \quad & \xi_c(Q_m) = m^3 2^m. \\ (iv) \quad & \xi_c(\Pi_m) = \begin{cases} 9m(m+2) & \text{when } m \text{ is even} \\ 9m(m+1) & \text{when } m \text{ is odd.} \end{cases} \\ (v) \quad & \xi_c(A_m) = \begin{cases} 16m^2 & \text{when } m \text{ is even} \\ 16m(m+1) & \text{when } m \text{ is odd.} \end{cases} \end{aligned}$$

The following results can also be computed from straight forward calculations.

Proposition 2.2. *Let W_n and B_n denote pyramid and bipyramid with $n(\geq 3)$ -gonal base, then $\xi_c(W_n) = 2n^2 + 5n$ and $\xi_c(B_n) = 4n^2 + 32n$.*

Proposition 2.3. Let K_{m_1, m_2, \dots, m_n} denotes the complete n -partite graph with $|V| = m_1 + m_2 + \dots + m_n$ number of vertices ($m_i \geq 2, i = 1, 2, \dots, n$), then

$$\xi_c(K_{m_1, m_2, \dots, m_n}) = 2 \sum_{i=1}^n m_i \left(\sum_{j=1, j \neq i}^n m_j (|V| - m_j) \right).$$

Proposition 2.4. Let $S_n = K_{1, n-1}$ be a star graph with $n (\geq 3)$ vertices, then $\xi_c(S_n) = 2n^2 - 3n + 1$.

We recall two fundamental indices, namely, the first and second Zagreb indices ($M_1(G)$ and $M_2(G)$) introduced by Gutman and Trinajstić [11], are two of the oldest and most studied vertex degree based topological indices which are defined respectively as the sum of squares of the degrees of the vertices, and sum of product of the degrees of the adjacent vertices of a graph. Now first we calculate some upper and lower bounds of modified eccentric connectivity index in terms of some graph parameters such as maximum vertex degree (Δ), minimum vertex degree (δ), radius (r), diameter (d) etc.

Theorem 2.5. Let $G = (V, E)$ be a simple connected graph with radius r and diameter d , then

- (i) $r \leq \frac{\xi_c(G)}{M_1(G)} \leq d$ and both hold with equality if and only if all the vertices of G are of same eccentricity.
- (ii) $r\delta^2 \leq \xi_c(G) \leq d\Delta^2$ and both hold with equality if and only if G is regular and all the vertices of G are of same eccentricity.
- (iii) $\delta^2 \leq \frac{\xi_c(G)}{\theta(G)} \leq \Delta^2$ and both hold with equality if and only if G is regular.

Proof. (i) Since we have, for any $v \in V(G), r \leq \varepsilon(v) \leq d$ and $\sum_{v \in V} \delta(G) = M_1(G)$, from (1.1), we get the desired result. Obviously, in this relation equality holds if and only if $r = \varepsilon(v) = d$ for all $v \in V$.

(ii) Since, for any $v \in V, \delta \leq \deg(v) \leq \Delta$ and $r \leq \varepsilon(v) \leq d$, so from (1.1) the desired result follows. Clearly, the equality holds if and only if all the vertices are of same degree and eccentricity.

(iii) Since, for any $v \in V, \delta \leq \deg(v) \leq \Delta$, so we have $\delta^2 \leq \delta(v) \leq \Delta^2$. Thus from the definition of modified eccentric connectivity index we have, $\Delta^2\theta(G) \leq \xi_c(G) \leq \delta^2\theta(G)$, with equality when G is a regular graph.

□

2.1. Upper bounds. In the following, we present some upper bounds for modified eccentric connectivity index of connected graphs.

Theorem 2.6. Let G be a simple connected graph, then

$$\xi_c(G) \leq \{2m - \delta(n - 1)\}\theta(G) + (\delta - 1)\xi^c(G)$$

and equality holds if and only if G is a regular graph.

Proof. We have, for any $v \in V(G)$, $\delta(v) \leq 2m - \deg(v) - (n - 1 - \deg(v))\delta$, $\xi_c(G) = \sum_{v \in V(G)} \delta(G)\varepsilon(v) \leq \sum_{v \in V(G)} \{2m - \deg(v) - (n - 1 - \deg(v))\delta\}\varepsilon(v)$
 $= 2m \sum_{v \in V(G)} \varepsilon(v) - \sum_{v \in V(G)} \deg(v)\varepsilon(v) - (n-1)\delta \sum_{v \in V(G)} \varepsilon(v) + \delta \sum_{v \in V(G)} \deg(v)\varepsilon(v)$, from where the desired result follows. Clearly in the above relation, equality holds if and only if G is a regular graph. \square

Corollary 2.7. *Let G be a simple connected graph, then*

$$\xi_c(G) \leq \{2m - \delta(n - 1)\} (n^2 - 2m) + (2mn - M_1(G))(\delta - 1)$$

with equality if and only if $G \cong K_n$.

Proof. Since, for any $v \in V(G)$, $\varepsilon(v) \leq n - \deg(v)$, with equality achieved for $G \cong K_n - je$ for $j = 1, 2, \dots, \lfloor n/2 \rfloor$ or $G \cong P_n$ [12], we have

$$\xi^c(G) = \sum_{v \in V(G)} \deg(v)\varepsilon(v) \leq \sum_{v \in V(G)} \deg(v)(n - \deg(v)) = 2nm - M_1(G) \text{ and } \theta(G) = \sum_{v \in V(G)} \varepsilon(v) \leq n^2 - 2m.$$

So, from the last theorem the desired result follows. \square

Analogues to Zagreb indices, Ghorbani and Hosseinzadeh [9] and Vukičević and Graovac [17] defined the Zagreb eccentricity indices by replacing degrees by eccentricity of the vertices, so that the first Zagreb eccentricity index ($E_1(G)$) is defined as sum of squares of the eccentricities of the vertices and the second Zagreb eccentricity index ($E_2(G)$) is equal to sum of product of the eccentricities of the adjacent vertices (see [4, 7, 17]). Now we find bounds of the modified eccentric connectivity index using these indices.

Theorem 2.8. *Let G be a simple connected graph, then*

$$\xi_c(G) \leq \sqrt{(\Delta^2 + \delta^2)M_1(G)E_1(G) - n\Delta^2\delta^2E_1(G)}$$

with equality holds if and only if all the vertices are of same degree and eccentricity.

Proof. From Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^n x_i y_i \leq \sqrt{\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2}.$$

Now putting $x_i = \delta(v_i)$ and $y_i = \varepsilon(v_i)$ for $i = 1, 2, \dots, n$, we have

$$(2.1) \quad \xi_c(G) = \sum_{i=1}^n \delta(v_i)\varepsilon(v_i) \leq \sqrt{\sum_{i=1}^n \delta(v_i)^2 \sum_{i=1}^n \varepsilon(v_i)^2} \leq \sqrt{E_1(G) \sum_{i=1}^n \delta(v_i)^2}$$

with equality if and only if all the vertices of G are of same degree and eccentricity.

Now, using the following Diaz-Metcalf inequality [1], we have if a_i and b_i , $i = 1, 2, \dots, n$ are real numbers such that $ma_i \leq b_i \leq Ma_i$ for $i = 1, 2, \dots, n$, then

$$(2.2) \quad \sum_{i=1}^n b_i^2 + mM \sum_{i=1}^n a_i^2 \leq (m+M) \sum_{i=1}^n a_i b_i.$$

In the above relation, equality holds if and only if $b_i = ma_i$ or $b_i = Ma_i$ for every $i = 1, 2, \dots, n$. By setting $b_i = \delta(v_i)$ and $a_i = 1$, for $i = 1, 2, \dots, n$, from above inequality we get

$$\sum_{i=1}^n \delta(v_i)^2 + mM \sum_{i=1}^n 1^2 \leq (m+M) \sum_{i=1}^n \delta(v_i).$$

Since $\delta^2 \leq \delta(v_i) \leq \Delta^2$ we have $m = \delta^2$ and $M = \Delta^2$, so that from above we get

$$\sum_{i=1}^n \delta(v_i)^2 \leq (\Delta^2 + \delta^2)M_1(G) - n\Delta^2\delta^2$$

with the equality if and only if $\delta(v_i) = \delta^2 = \Delta^2$ for $i = 1, 2, \dots, n$, i.e., G is regular graph. Thus from (2.1) the desired result follows. Obviously in this theorem equality holds if and only if all the vertices are of same degree and eccentricity. \square

Theorem 2.9. *Let G be a simple connected graph, then*

$$\xi_c(G) \leq nM_1(G) - 2M_2(G),$$

with equality holds if and only if $G \cong K_n - je$ for $j = 1, 2, \dots, \lfloor n/2 \rfloor$ or $G \cong P_n$.

Proof. We have for any $v \in V(G)$, $\varepsilon(v) \leq n - \deg(v)$, with equality achieved for $G \cong K_n - je$ for $j = 1, 2, \dots, \lfloor n/2 \rfloor$ or $G \cong P_n$ [12]. So from the definition of modified eccentric connectivity index

$$\xi_c(G) = \sum_{v \in V(G)} \delta(v)\varepsilon(v) \leq \sum_{v \in V(G)} \delta(v)(n - \deg(v)) = n \sum_{v \in V(G)} \delta(v) - \sum_{v \in V(G)} \deg(v)\delta(v).$$

Now, $\sum_{v \in V(G)} \deg(v)\delta(v) = 2M_2(G)$ and the desired result follows from above. \square

2.2. Lower bounds. Now we find some lower bounds of modified eccentric connectivity index in terms of maximum vertex degree (Δ), minimum vertex degree (δ), radius (r), diameter (d), total eccentricity ($\theta(G)$), the first Zagreb index ($M_1(G)$), the eccentric connectivity index ($\xi^c(G)$).

Theorem 2.10. *Let G be a simple connected graph, then*

- (i) $\xi_c(G) \geq M_1(G)$ where, $M_1(G)$ is the first Zagreb index of G and equality holds if and only if $\varepsilon(v) = 1$ for all $v \in V(G)$ i.e., $G \cong K_n$.
- (ii) $\xi_c(G) \geq \xi^c(G)$ where, $\xi^c(G)$ is the eccentric connectivity index of G and equality holds if and only if $G \cong P_3$.

Proof. (i) Since $\varepsilon(v) \geq 1$ for all $v \in V(G)$, from the definition of modified eccentric connectivity index the desired result follows with equality when $\varepsilon(v) = 1$ i.e., $G \cong K_n$.

- (ii) Again since $\delta(v) \geq \deg(v)$ for all $v \in V(G)$, we have from definition of modified eccentric connectivity index $\xi_c(G) = \sum_{v \in V(G)} \delta(v)\varepsilon(v) \geq \sum_{v \in V(G)} \deg(v)\varepsilon(v) = \xi^c(G)$. Clearly in this relation equality holds if and only if G is a path of length two. □

Theorem 2.11. *Let G be a simple connected graph, then*

$$\xi_c(G) \geq \frac{1}{d\Delta^2 + r\delta^2} \left[\Delta^2\delta^2 E_1(G) + \frac{rd}{n} M_1(G)^2 \right]$$

and it holds with equality if and only if all the vertices of G are of same degree and eccentricity.

Proof. Putting $b_i = \delta(v_i)$ and $a_i = \varepsilon(v_i)$, for $i = 1, 2, \dots, n$, in the Diaz-Metcalf inequality (2.2), we have

$$\sum_{i=1}^n \delta(v_i)^2 + mM \sum_{i=1}^n \varepsilon(v_i)^2 \leq (m+M) \sum_{i=1}^n \delta(v_i)\varepsilon(v_i),$$

$$(2.3) \quad \text{i.e., } (m+M)\xi_c(G) \geq \sum_{i=1}^n \delta(v_i)^2 + mM E_1(G).$$

Now since $\frac{\delta^2}{d} \leq \frac{\delta(v_i)}{\varepsilon(v_i)} \leq \frac{\Delta^2}{r}$, from the equality condition of Diaz-Metcalf inequality (2.2), we have $m = \frac{\delta^2}{d}$ and $M = \frac{\Delta^2}{r}$.

Again from Cauchy-Schwartz inequality we have,

$$(2.4) \quad n \sum_{i=1}^n \delta(v_i)^2 \geq \left[\sum_{i=1}^n \delta(v_i) \right]^2 = M_1(G)^2$$

with equality if and only if all the vertices of G are of same degree, so that from (2.3) we have

$$\left(\frac{\delta^2}{d} + \frac{\Delta^2}{r} \right) \xi_c(G) \geq \frac{1}{n} M_1(G)^2 + \frac{\Delta^2\delta^2}{rd} E_1(G),$$

from where the desired result follows. Clearly equality achieved if and only if all the vertices are of same degree and eccentricity. □

Theorem 2.12. *Let G be a simple connected graph, then*

$$\xi_c(G) \geq \sqrt{\frac{1}{n} M_1(G) E_1(G) - \frac{n^2}{4} (d\Delta^2 + r\delta^2)}.$$

Proof. Using the Ozeki's inequality [16] we have, if a_1, a_2, \dots, a_m and b_1, b_2, \dots, b_m be positive real numbers such that for $1 \leq i \leq n$, $m_1 \leq a_i \leq M_1$ and $m_2 \leq b_i \leq M_2$ hold, then

$$\left\{ \sum_{i=1}^n a_i^2 \right\} \left\{ \sum_{i=1}^n b_i^2 \right\} - \left\{ \sum_{i=1}^n a_i b_i \right\}^2 \leq \frac{n^2}{4} \{M_1 M_2 - m_1 m_2\}^2.$$

Now putting $a_i = \varepsilon(v_i)$ and $b_i = \delta(v_i)$ for $i = 1, 2, \dots, n$, so that $m_1 = r$, $M_1 = d$ and $m_2 = \delta^2$, $M_2 = \Delta^2$, we have

$$\left\{ \sum_{i=1}^n \varepsilon(v_i)^2 \right\} \left\{ \sum_{i=1}^n \delta(v_i)^2 \right\} - \left\{ \sum_{i=1}^n \varepsilon(v_i)\delta(v_i) \right\}^2 \leq \frac{n^2}{4} \{d\Delta^2 + r\delta^2\}^2,$$

i.e., $E_1(G) \sum_{i=1}^n \delta(v_i)^2 - \xi_c(G)^2 \leq \frac{n^2}{4} \{d\Delta^2 + r\delta^2\}^2$.

Now using (2.3) we have

$$\xi_c(G)^2 \geq \frac{1}{n} E_1(G) M_1(G) - \frac{n^2}{4} \{d\Delta^2 + r\delta^2\}^2$$

which is our desired result. □

Theorem 2.13. *Let G be a simple connected graph, then*

$$\xi_c(G) \geq \frac{2}{(n-1)} M_2(G)$$

and equality holds if and only if $G \cong K_n$.

Proof. For any $v \in V(G)$, we have $\varepsilon(v) \geq \frac{D(v)}{(n-1)}$, with equality if and only if $G \cong K_n$. So from the definition of modified eccentric connectivity index

$$(2.5) \quad \xi_c(G) = \sum_{v \in V(G)} \delta(v)\varepsilon(v) \geq \sum_{v \in V(G)} \delta(v) \frac{D(v)}{(n-1)}.$$

Now since for any $v \in V(G)$, $D(v) \geq \deg(v)$, we have from (2.2)

$$(2.6) \quad \xi_c(G) \geq \sum_{v \in V(G)} \delta(v) \frac{\deg(v)}{(n-1)} = \frac{1}{(n-1)} \sum_{v \in V(G)} \delta(v) \deg(v)$$

from where the desired result follows. Clearly, in the above relation equality holds if and only if $G \cong K_n$. □

Theorem 2.14. *Let G be a simple connected graph, then*

$$\xi_c(G) \geq 2M_1(G) - \frac{2M_2(G)}{(n-1)}$$

and it holds with equality if and only if G is a path of length one.

Proof. Since, $D(v) \geq 2n - 2 - \deg(v)$ for all $v \in V(G)$, with equality if and only if $G \cong K_{1,n-1}$, so from (2.5) we get,

$$\begin{aligned} \xi_c(G) &\geq \frac{1}{(n-1)} \sum_{v \in V(G)} \delta(v)(2n - 2 - \deg(v)) \\ &= \frac{1}{(n-1)} \left[2(n-1) \sum_{v \in V(G)} \delta(v) - \sum_{v \in V(G)} \delta(v) \deg(v) \right] \end{aligned}$$

from where the desired result follows. Since the equality (2.6) holds if and only if $G \cong K_n$, the equality holds in this result if and only if G is a path of length one which is a complete graph as well as complete bipartite graph. □

Theorem 2.15. *Let G be a simple connected graph, then*

$$\xi_c(G) \geq \delta^{\frac{\delta}{\Delta}} \xi^c(G)$$

with equality if and only if G is a path of length one.

Proof. Using the relationship between arithmetic and geometric mean, we have

$$\frac{1}{\deg(v)} \delta(v) = \frac{1}{\deg(v)} \sum_{(u,v) \in E(G)} \deg(u) \geq \left[\prod_{(u,v) \in E(G)} \deg(u) \right]^{\frac{1}{\deg(v)}}$$

i.e., $\delta(v) \geq \deg(v) \delta^{\frac{\delta}{\Delta}}$.

Thus from the definition of modified eccentric connectivity index we have,

$$\xi_c(G) = \sum_{v \in V(G)} \delta(v) \varepsilon(v) \geq \delta^{\frac{\delta}{\Delta}} \sum_{v \in V(G)} \deg(v) \varepsilon(v)$$

which is our desired result. Clearly, equality holds if and only if all the vertices of G are of same degree. \square

Recall that the Wiener index of a connected graph G (see [22, 18]) is denoted by $W(G)$ and is defined as

$$W(G) = \sum_{u,v \in V(G)} d(u,v) = \frac{1}{2} \sum_{v \in V(G)} D(v).$$

Theorem 2.16. *Let G be a simple connected graph, then*

$$\xi_c(G) \geq \frac{2\delta^2}{n-1} W(G)$$

with equality if and only if $G \cong K_n$.

Proof. Since for any $v \in V(G)$, $\deg(v) \geq \delta$, we have from (2.2)

$$\xi_c(G) \geq \frac{\delta^2}{(n-1)} \sum_{v \in V(G)} D(v) = \frac{2\delta^2}{(n-1)} W(G)$$

with equality if and only if $G \cong K_n$. \square

From the definition of Harary index [3, 21], it follows that $W(G) \geq H(G)$, with equality if and only if $G \cong K_n$. Then from the above theorem the following Corollary follows.

Corollary 2.17. *Let G be a simple connected graph, then*

$$\xi_c(G) \geq \frac{2\delta^2}{n-1} H(G)$$

with equality if and only if $G \cong K_n$.

We now give a Nordhaus-Gaddum type [15] result of modified eccentric connectivity index of connected graph.

Theorem 2.18. *Let G be a simple connected graph with $n \geq 4$ vertices, for which the complement \overline{G} is also connected, then*

$$\xi_c(G) + \xi_c(\overline{G}) \geq 2 [M_1(G) + M_1(\overline{G})]$$

and this holds with equality if and only if all the vertices of G are of eccentricity two.

Proof. Since both G and \overline{G} are connected graph and each has radius at least 2, from the definition of modified eccentric connectivity index the desired result follows. \square

Theorem 2.19. *Let G be a n -vertex simple connected graph with $n \geq 3$ vertices, then*

$$\xi_c(G) \geq (2n - 1)(n - 1)$$

with equality if and only if $G \cong S_n$.

Proof. A three vertex connected graph is either S_3 or K_3 . It may be easily checked that $\xi_c(S_3) = (2n - 1)(n - 1) \leq \xi_c(K_3)$. Let m be the number of edges of G and $k(0 \leq k \leq n)$ be the number of vertices of G of degree $n - 1$ and eccentricity one. So for these k vertices, $\delta(v) = 2m - (n - 1)$. Obviously the remaining $n - k$ vertices are of degree less than $n - 1$ and eccentricity two.

Thus, $\xi_c(G) \geq k \{2m - (n - 1)\} + 2 [M_1(G) - 2k \{2m - (n - 1)\}]$, with equality if and only if all the $n - k$ vertices are of degree less than $n - 1$ and of eccentricity 2.

If $k \geq 1$, then all the vertices of G except one vertex are of degree at least k . So we can write, $2m \geq k(n - 1) + k(n - k)$ and $M_1(G) \geq (n - 1)^2 + k^2(n - 1)$. Thus,

$$\xi_c(G) \geq 2(n - 1)(n - 1 + k^2) - k \{k(n - 1) + k(n - k) - (n - 1)\},$$

i.e., $\xi_c(G) \geq k^3 + k(n - 2) + 2(n - 1)^2$.

Clearly, the function $f(x) = x^3 + x(n - 2) + 2(n - 1)^2$ with $1 \leq x \leq n$ attains the minimum value for $x = 1$, where $f(1) = (n - 1)(2n - 1)$. Hence $\xi_c(G) \geq f(1) = (2n - 1)(n - 1)$ with equality if and only if $k = 1$ and $m = n - 1$ i.e., $G \cong S_n$. \square

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