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On the Decomposition of Total Graphs

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Abstract

Obtaining a graph from any given graph is a popular area of research in Graph Theory. Concept of Total Graph falls under this category. All the vertex-vertex adjacency, vertex-edge incidence and edge-edge incidence relations are considered in the formation of the Total Graph. For a finite simple connected graph G, T(G) can be decomposed into G and complete subgraphs of order equal to the degrees of each of the vertices in G. Also, T(G) can be decomposed into disjoint union of L(G) and q copies of C_3, where q is the size of G.

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1. Introduction

We consider a graph G(p, q) with p vertices and q edges which is simple, connected, undirected and finite. Here, p and q are respectively called the order and size of G. Let v be a vertex of G. The number of edges incident with v is known as the degree of v, denoted by $\deg_G(v)$, or merely by deg(v).[Chartrand, 2006] If the degrees of the vertices of a graph G are listed in a non-increasing sequence S, then S is called the degree sequence of G. For a graph G, obtaining edge disjoint sub-graphs (i.e. intersection of the edge set of all the sub-graphs is empty) whose union is the actual graph G is called decomposition of the given graph G. Line Graph, L(G), of undirected graph G is a graph that represents the adjacencies between the edges of G. Given a graph G, each vertex of L(G) represents an edge of G and two vertices of L(G) are adjacent if and only if their corresponding edges are adjacent in G. An Incidence Graph, I(G), is a graph whose vertices represent vertices and edges in G. Two vertices in I(G) are adjacent if and only if there is a vertex-edge incidence in G. Total Graph of a graph G, denoted by T(G), is a graph whose vertices are represented by each vertex and each edge of G. There is an edge between two vertices in T(G) if and only if there is edge-edge adjacency or edge-vertex incidence or vertex-vertex adjacency in G. [West, 2002 and Harary, 2001]

We know that T(G) is isomorphic to the square of the subdivision graph S(G).
i.e. $\mathbf{T(G) \approx [S(G)]^2}$.[Harary, 2001]
But we also know that S(G) ≈I(G).
Hence, T(G) is isomorphic to the square of the incidence graph I(G).
i.e. $\mathbf{T(G) \approx [I(G)]^2}$
From the definition of total graph we can also define the total graph as the disjoint union of given graph, line graph and incidence graph.
i.e. $\mathbf{T(G) = G} \cup \boldsymbol{L(G)} \cup \boldsymbol{I(G)}$
This is possible because in T(G) vertex-vertex adjacency will give us G itself, edge-edge adjacency gives us line graph of G, denoted by L(G) and vertex-edge incidence will give

us incidence graph of G, denoted by I(G). From the definition of total graph G, it is obvious that L(G) and I(G) in T(G) are disjoint.

2. Decomposition of T(G) into G and K_n's

Let Kn denote a complete graph of n vertices. Every edge in G becomes a K_3 in T(G). If we explore this phenomenon, we obtain the following result.

Theorem 2.1. Let G be an undirected simple finite graph. Total Graph of G can be decomposed into G and K_{d_i+1} 's, where d_i's are degrees of each of the vertices in G. i.e. T(G) = G U K_{d1+1} U K_{d2+1} U …….. U K_{dn+1}, where d_i's are degrees of each vertex in G.

Proof: Since T(G) is the total graph of G, every vertex in T(G) is represented by either a vertex or an edge in G. Two vertices in T(G) are adjacent if and only if there is a corresponding vertex-vertex adjacency or edge-edge adjacency or an edge-vertex incidence in G. Now, the vertex-vertex adjacency in G will give exactly the same copy of G in T(G). We also know that for each vertex–edge incidence and edge–edge adjacency in G, there exists an edge in T(G).
Let v_1 be an arbitrary vertex in G with degree d_1.
So v_1 is incident with d_1 edges.
Let $e_1, e_2 \ldots\ldots e_{d1}$ be these edges.
i.e., all these e_i's are incident with v_1. Hence in T(G), a vertex corresponding to v_1 is adjacent to all vertices corresponding to $e_{i's}$.
Since in G, all e_i's are incident to v_1, obviously all e_i's are adjacent with each other.
Hence all e_i's will form a complete graph with d_i vertices in T(G).
But all e_i's are incident with v_1 and hence with the addition of the corresponding vertex in T(G) to the already formed complete graph, the new complete graph is with d_1+1 vertices.
i.e. K_{d1+1} is formed in T(G).
Since v_1 is arbitrary, it is true for all vertices.
Now we have to show that all such complete graphs are disjoint.
Let w be an edge common to K_{d1+1} and K_{d2+1} in T(G).
i.e., w is there in K_{d1+1} and w is also there in K_{d2+1}.
Hence the end vertices of w must be in both K_{d1+1} and K_{d2+1}.
Let w=e_1e_2.
We know that e_1 and e_2 are adjacent in T(G) since their corresponding edges are incident with some v_1 in G.
Hence they are adjacent in K_{d1+1}.
We know that since w is also in K_{d2+1} and the corresponding vertices of e_1 and e_2 are adjacent in G, which means they are incident with another vertex other than v_1.
Let it be v_2.
Therefore e_1 and e_2 are incident with v_1 and v_2.
But this will lead to a multiple edge in G.
It is a contradiction, since G is a simple graph.
Hence all the complete graphs in T(G) are disjoint.
Hence we can decompose T(G) into disjoint union of G and p complete graphs with di+1 vertices, where di is the degree of each of the p vertices in G.
Hence the proof.

Corollary 2.1.1. Let K_n be a complete graph with n vertices. Then $T(K_n)$ =
$\bigcup_{i=1}^{n+1} K_{n_i}$ K_{ni}'s are copies of K_n.

Proof: From **Theorem 2.1** we get, $T(G) = G \cup K_{d1+1} \cup K_{d2+1} \cup \ldots\ldots\ldots \cup K_{dn+1}$, where d_i's are degrees of the vertices in K_n.
There are n vertices in K_n all of degree n-1.
i.e. $d_i = n-1$
Hence
$T(G) = G \cup K_{n-1+1} \cup K_{n-1+1} \cup \ldots\ldots\ldots \cup K_{n-1+1}$
$T(G) = G \cup K_n \cup K_n \cup \ldots\ldots\ldots \cup K_n$.
So T(G) can be decomposed into G and union of n copies K_n.
Here G is K_n.
Therefore T(G) can be decomposed into union of (n+1) K_n's.
i.e., $T(K_n) = \bigcup_{i=1}^{n+1} K_{n_i}$, where K_{ni}'s are copies of K_n.
Hence the proof.

3. Decomposition of T(G) into L(G) and C_3's

We know that total graph of any graph is the disjoint union of line graph, incidence graph of the given graph and the given graph itself. The edge-vertex incidence of each edge in G is producing a C_3 in T(G). It is seen that number of these C_3's can be found out. It is described in the next theorem.

Theorem 3.1. Let G(p,q) be a simple undirected finite simple graph. Then T(G) can be decomposed into L(G) and q copies of C_3.

Proof: Let G(p,q) be the given Graph. The total graph of G is the disjoint union of G and the line graph of G and incidence graph of G.
i.e. $T(G) = G \cup L(G) \cup I(G)$ where G, L(G) and I(G) are disjoint.
Clearly, T(G) contains L(G) .
So when we remove L(G) from T(G) what is remaining T(G) is GUI(G).
Let $e = uv$ be an edge in G.
Hence e will become a vertex in I(G) and will be incident with u and v.
Therefore eu and ev will be two distinct edges in I(G).
Evidently in GU I(G), e-u-v-e will form C_3.
Since e is arbitrary, for each edge in G we get a new copy of C_3.
Since G contains q edges, we get q copies of C_3.
Thus T(G) can be decomposed into L(G) and q copies of C_3.
Hence the proof.

Corollary 3.1.1. Let C_n be the cycle with n vertices, then $T(C_n)$ can be decomposed into C_n and n copies of C_3.

Proof: The proof is direct from the Theorem 3.1.
Here, G is Cn. Cn has n edges.
Also, $L(C_n) = C_n$.
Hence from the above theorem we can conclude that $T(C_n)$ can be decomposed into C_n and n copies of C_3.
Hence the proof.

4. Conclusion

In this paper we had concentrated on decomposition of total graphs. The results that we discussed in decomposition of total graph are $T(G) = GUK_{d1}UK_{d2}U \ldots\ldots\ldots U\ K_{dn.}$ and $T(G(p,q)) = L(G)U\mathit{q}C_3$. There is a lot of scope for the further study of decomposition of total graphs of some graph operations like Cartesian product, tensor product etc.

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