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A Global R-linear Convergence Algorithm for the Generalized Linear Complementarity Problem Over a Closed Convex Cone

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Abstract. In this paper, we propose a iterative method for solving the generalized linear complementarity problem (GLCP) over a closed convex cone, its global convergence is proved under mild conditions. Furthermore, the error bound for GLCP is also given under suitable conditions, based on this, we prove that the method has R-linear convergence rate.

Keywords: GLCP, iterative method, error bound, global convergence, R-linear convergence rate.

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1 Introduction

Let $F(x) = Mx + p, G(x) = Nx + q$, where $M, N \in R^{m\times n}$, $p, q \in R^m$. the generalized linear complementarity problem, abbreviated as GLCP, is to find vector $x^* \in R^n$ such that

$$F(x^*) \in \mathcal{K}, \quad G(x^*) \in \mathcal{K}^0, \quad F(x^*)^\top G(x^*) = 0, \tag{1}$$

where $\mathcal{K}$ be a nonempty closed convex cone in R^m and $\mathcal{K}^\circ$ is its dual cone,i.e., $\mathcal{K}^\circ = \{u \in R^m \mid u^\top v \geq 0, \forall v \in \mathcal{K}\}$. We denote the solution set of the GLCP by X^* and assume that it is nonempty throughout this paper.

The GLCP is a direct generalization of the classical linear complementarity problem (LCP) which finds applications in engineering, economics, finance, and robust optimization operations research (Ref.[1]).

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For example, the GLCP plays a significant role in contact mechanics problems(such as a dynamic rigid-body model, a discretized large displacement frictional contact problem), structural mechanics problems, obstacle problems mathematical physics, elastohydrodynamic lubrication problems, traffic equilibrium problems(such as a path-based formulation problem, a multicommodity formulation problem, network design problems),etc([1]), and also plays a significant role in economics, such as the supply chain network equilibrium model(Refs.[2, 3, 4]). Up to now, the issues of numerical methods and existence of the solution for the problem were discussed in the literature (e.g., Ref. [5]).

In recent years, many effective methods have been proposed for solving GLCP which $\mathcal{K}$ is a polyhedral cone in R^m, that is, there exists $A \in R^{s\times m}, B \in R^{t\times m}$, such that $\mathcal{K} = \{v \in R^m \mid Av \geq 0,\ Bv = 0\}$, the basic idea of these methods is to reformulate the problem as an unconstrained or simply constrained optimization problem ([6, 7, 8, 9, 10, 11]), the condition which the nonsingularity of Jacobian at a solution guarantees that the L-M method for GLCP has global convergence ([9, 8]), or it which the mapping G is monotone with respect to F guarantees that method be proposed by Sun also has global convergence([10, 11]). This motivates us to consider the new method for the GLCP under mild conditions. So, in this paper, we propose the new iterative method which is different from the algorithms listed above to solve GLCP, and we establish the global convergence under mild condition. Furthermore, we also present a error bound for GLCP under the suitable conditions, based on this, the linear convergence rate analysis of the proposed algorithm also is presented in this paper. Compared with the existing solution methods in [9, 8, 10, 11], the conditions guaranteed for convergence are weaker.

Some notations used in this paper are in order. R^n be a real Euclidean space, whose inner product and the Euclidean 2-norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. We denote the pesudo-inverse of a matrix M by M^+.

2 Preliminary

In this section, we will establish an equivalent reformulation of the GLCP, i.e., convert the GLCP into a variational inequality problem, and state some well known properties of the projection operator. Now, we give the following assumptions which is crucial to our method.

Assumption 2.1$\langle G(x), F(x) - F(x^*)\rangle \geq 0,\ \forall x \in R^n, x^* \in X^*$.

Remark 2.1 If G is $F-$ pseudomonotone, then we have assumption 2.1 holds([12]).

In the following, we give the equivalent reformulation of the GLCP.

Theorem 2.1 x^* is a solution of (1) if and only if x^* is a solution of the following problem

$$G(x^*)^\top((F(x) - F(x^*)) \geq 0, \quad \forall F(x) \in \mathcal{K}. \tag{2}$$

Proof. Suppose that x^* is a solution of (2). Since vector $0 \in \mathcal{K}$, by substituting $F(x) = 0$ into (2), we have $G(x^*)^\top F(x^*) \leq 0$. On the other hand, since $F(x^*) \in \mathcal{K}$, then $2F(x^*) \in \mathcal{K}$. By substituting $F(x) = 2F(x^*)$ into (2), we obtain $G(x^*)^\top F(x^*) \geq 0$. Consequently, $G(x^*)^\top F(x^*) = 0$. For any $F(x) \in \mathcal{K}$, we have $G(x^*)^\top F(x) = G(x^*)^\top [F(x) - F(x^*)] \geq 0$, i.e., $G(x^*) \in \mathcal{K}^\circ$. Thus, x^* is a solution of (1).

On the contrary, suppose that x^* is a solution of (1), since $G(x^*) \in \mathcal{K}^\circ$, for any $F(x) \in \mathcal{K}$, we have $G(x^*)^\top F(x) \geq 0$, combining $G(x^*)^\top F(x^*) = 0$, we have $G(x^*)^\top [F(x) - F(x^*)] \geq 0$, Thus, x^* is a solution of (2).

Now, we give the definition of projection operator and some relate properties. For nonempty closed convex set $\Omega \subset R^n$ and any vector $x \in R^n$, the orthogonal projection of x onto Ω, i.e., $argmin\{\|y - x\| | y \in \Omega\}$, is denoted by $P_\Omega(x)$.

Lemma 2.1 For any $u \in R^n, v \in \Omega$, then

$$\begin{aligned} &(i) \;\; \langle P_\Omega(u) - u, v - P_\Omega(u) \rangle \geq 0, \\ &(ii) \;\; \|P_\Omega(u) - P_\Omega(v)\| \leq \|u - v\|. \end{aligned}$$

From Theorem 2.1, one can prove that (2) is equivalent to the fixed-point problem, this result is due to Noor([13]). For convenience, throughout this paper, we define the projection residue vector

$$R(x, \rho) := F(x) - P_\mathcal{K}[F(x) - \rho G(x)], \quad \rho > 0.$$

Lemma 2.2 x^* is a solution of the GLCP if and only if $R(x^*, \rho) = 0$, for some $\rho > 0$.

Based on this fixed-point formulation, various projection type iterative method for solving variational inequalities have been suggested and analyzed, see[14, 13, 15].

To propose algorithm for solving the GLCP, we also need the following conclusion in [16].

Lemma 2.3 For the non-homogeneous linear equation system $Hy = b$. Then $y = H^+ b$ is unique least square solution with the minimum 2-norm, where H^+ is the pesudo-inverse of H.

3 Algorithm and Global Convergence

Now, we formally describe our method for solving the GLCP.

Algorithm 3.1

Step1 Choose $x^0 \in R^n$ such that $F(x^0) \in \mathcal{K}$, select any $0 < \sigma < \min\{1, \|NM^+\|^{-1}\}$, $\rho_{-1} = 1, 0 < \varphi < 2(1 - \sigma\|NM^+\|)/(1 - \sigma), \theta > 0$, set $k := 0$.

Step2 For $F(x^k) \in \mathcal{K}$, take $y^{k-1} \in R^n$ such that

$$F(y^{k-1}) = P_\mathcal{K}\{F(x^k) - \rho_{k-1}[NM^+ F(x^k) - NM^+ p + q]\}.$$

If $R(x^k,\rho_{k-1}) = F(x^k) - F(y^{k-1}) = 0$, then go to Step 3. Otherwise, Let $\rho_k = \theta\gamma^{m_k}$, where m_k being the smallest nonnegative integer m satisfying

$$\rho_k\|F(x^k) - F(x^k(\rho_k))\| \le \sigma\|R(x^k,\rho_k)\|, \tag{3}$$

where

$$F(x^k(\rho_k)) = P_{\mathcal{K}}\{F(x^k) - \rho_k[NM^+F(x^k) - NM^+p + q]\}. \tag{4}$$

Compute x^{k+1} by solving the following equation

$$F(x^{k+1}) = P_{\mathcal{K}}[F(x^k) + \varphi\alpha_k d_k],$$

where

$$d_k = -\{R(x^k,\rho_k) - \rho_k(NM^+F(x^k) - NM^+F(x^k(\rho_k)))\}, \tag{5}$$

$$\alpha_k = (1-\sigma)\|R(x^k,\rho_k)\|^2/\|d_k\|^2.$$

Step3 Let $x^{k+1} = M^+(F(x^{k+1}) - p)$, stop.

Remark 3.1 In algorithm 3.1, several implicit equation of F needn't be solved at each iteration. ρ_k, α_k are said to be predictor stepsizes and the corrector stepsizes, respectively.

Remark 3.2 we recall the searching direction $-\{\eta_k R(u^k,\rho)+\eta_k T(u^k)+\rho T(v^k\}$ appear in [14] for solving general variational inequalities by Noor, Wang and Xiu, and differ from the direction in our algorithm.

Now, we discuss the feasibility of stepsize rule of (3).

Lemma 3.1 If x^k is not a solution of GLCP, then for any $\sigma \in (0,1)$, there exists $\hat{\rho}(u^k) \in (0,1]$, for any $\rho \in (0, \hat{\rho}(x^k)]$, we have

$$\rho\|F(x^k) - F(x^k(\rho))\| \le \sigma\|R(x^k,\rho)\|. \tag{6}$$

where $x^k \in R^n$ and $F(x^k) \in \mathcal{K}$, $F(x^k(\rho))$ be defined in (4).

Proof. Assume that there exists $\sigma \in (0,1)$, for any $0 < \hat{\rho} \le 1$, there exists $0 < \rho \le \hat{\rho}$ such that

$$\rho\|F(x^k) - F(y(\rho))\| > \sigma\|R(x^k,\rho)\|. \tag{7}$$

Let $\hat{\rho}$ goes to 0, then we have that ρ tends to 0, furthermore, for any $\varepsilon > 0$, we take $\bar{\delta} = \varepsilon$ such that

$$\begin{aligned} \|F(x^k) - F(x^k(\rho))\| &= \|F(x^k) - P_{\mathcal{K}}(F(x^k) - \rho G(x^k))\| \\ &= \|P_{\mathcal{K}}F(x^k) - P_{\mathcal{K}}(F(x^k) - \rho G(x^k))\| \\ &\le \|G(x^k)\|\rho \le \bar{\delta}, \end{aligned} \tag{8}$$

combining this with (7), using continuity of $G,\ F$, we have

$$\sigma\|R(x^k,\rho)\| < \rho\|\|F(x^k) - F(x^k(\rho))\| \le \rho\varepsilon,$$

combining Lemma 2.2, we know that $x^k \in X^*$, it contradicts that x^k isn't a solution of GLCP.

To establish the global (linear) convergence of Algorithm 3.1, we also need the following technical lemmas.

Lemma 3.2 Under Assumption 2.1, for given $x^* \in X^*$, then

$$\langle F(x^k) - F(x^*), -d_k \rangle \geq (1 - \sigma \|NM^+\|)\|R(x^k, \rho_k)\|^2.$$

Proof. From the iterative procedure, for any positive integer k, we know that $F(x^k),\ F(x^k(\rho_k)) \in \mathcal{K}$. By $x^* \in X^*$, Lemma 2.1(i), we have

$$\langle [F(x^k) - \rho_k G(x^k)] - F(x^k(\rho_k)), F(x^k(\rho_k)) - F(x^*) \rangle \geq 0,$$

combining this with definition of $R(u, \rho)$, we know that

$$\langle R(x^k, \rho_k) - \rho_k G(x^k), F(x^k) - F(x^*) - R(x^k, \rho_k) \rangle \geq 0,$$

we obtain

$$\langle R(x^k, \rho_k), F(x^k) - F(x^*) \rangle - \|R(x^k, \rho_k)\|^2 - \langle \rho_k G(x^k), F(x^k) - F(x^*) \rangle + \langle \rho_k G(x^k), R(x^k, \rho_k) \rangle \geq 0. \quad (9)$$

From Lemma 2.3, we have $x^k = M^+F(x^k) - M^+p$. Thus, we also have $G(x^k) = Nx^k + q = NM^+F(x^k) - NM^+p + q$. Combining this with (9), we obtain

$$\begin{aligned} &\langle R(x^k, \rho_k) - \rho_k [NM^+F(x^k) - NM^+p + q], F(x^k) - F(x^*) \rangle \\ &\quad \geq \|R(x^k, \rho_k)\|^2 - \langle \rho_k [NM^+F(x^k) - NM^+p + q], R(x^k, \rho_k) \rangle. \end{aligned} \quad (10)$$

On the other hand, by Assumption 2.1, we have

$$\langle G(x^k(\rho_k)), F(x^k(\rho_k)) - F(x^*) \rangle \geq 0, \quad (11)$$

combining definition of $F(x^k(\rho_k))$ in Algorithm 3.1 with definition of $R(u, \rho)$, we know that $F(x^k(\rho_k)) = F(x^k) - R(x^k, \rho_k)$. Thus, substituting $F(x^k(\rho_k))$ in (11) with $F(x^k) - R(x^k, \rho_k)$, we get

$$\begin{aligned} 0 \ &\leq \langle F(x^k(\rho_k)) - F(x^*), G(x^k(\rho_k)) \rangle \\ &= \langle F(x^k) - R(x^k, \rho_k) - F(x^*), G(x^k(\rho_k)) \rangle \\ &= \langle F(x^k) - F(x^*), G(x^k(\rho_k)) \rangle - \langle R(x^k, \rho_k), G(x^k(\rho_k)) \rangle, \end{aligned}$$

i.e.,

$$\langle F(x^k) - F(x^*), [NM^+F(x^k(\rho_k)) - NM^+p + q] \rangle \geq \langle R(x^k, \rho_k), [NM^+F(x^k(\rho_k)) - NM^+p + q] \rangle. \quad (12)$$

Using definition of d_k in Algorithm 3.1, we have

$$\begin{aligned}\langle F(x^k)-F(x^*),-d_k\rangle \quad &= \langle F(x^k)-F(x^*),R(x^k,\rho_k)+\rho_k\{[NM^+F(x^k(\rho_k))-NM^+p+q]\\ &\quad -[NM^+F(x^k)-NM^+p+q]\}\rangle\\ &= \langle F(x^k)-F(x^*),R(x^k,\rho_k)-\rho_k[NM^+F(x^k)-NM^+p+q]\rangle\\ &\quad +\langle F(x^k)-F(x^*),\rho_k[NM^+F(x^k(\rho_k))-NM^+p+q]\rangle\\ &\geq \|R(x^k,\rho_k)\|^2-\langle\rho_k[NM^+F(x^k)-NM^+p+q],R(x^k,\rho_k)\rangle\\ &\quad +\langle\rho_k R(x^k,\rho_k),[NM^+F(x^k(\rho_k))-NM^+p+q]\rangle\\ &= \|R(x^k,\rho_k)\|^2-\rho_k\langle NM^+F(x^k)-NM^+F(x^k(\rho_k)),R(x^k,\rho_k)\rangle\\ &\geq \|R(x^k,\rho_k)\|^2-\rho_k\|NM^+F(x^k)-NM^+F(x^k(\rho_k))\|\|R(x^k,\rho_k)\|\\ &\geq (1-\sigma\|NM^+\|)\|R(x^k,\rho_k)\|^2.\end{aligned}$$

where the first inequality is by (10) and (12), the second inequality is based on Cauchy-Schwarz inequality, the third inequality is by (3).

Lemma 3.3 Under Assumption 2.1, the sequence $\{\alpha_k\}$ and $\{\rho_k\}$ generated by algorithm 3.1 both have a uniformly positive bound from below, respectively.

Proof. Firstly, we shall show that α_k have a uniformly positive bound from below. Using representation of d_k and (3), we know that

$$\begin{aligned}\|d_k\|^2 \quad &\leq \quad 2\|R(x^k,\rho_k)\|^2+2\rho_k^2\|NM^+F(x^k)-NM^+F(y^k)\|^2\\ &\leq \quad 2(1+(\sigma\|NM^+\|)^2)\|R(x^k,\rho_k)\|^2.\end{aligned}$$

By representation of α_k again in Algorithm 3.1, we have that there exists a constant $\eta>0$ such that

$$\begin{aligned}\alpha_k \quad &= (1-\sigma)\|R(x^k,\rho_k)\|^2/\|d_k\|^2\\ &\geq \tfrac{1-\sigma}{2(1+(\sigma\|NM^+\|)^2)} =: \eta.\end{aligned}$$

In the following, we also prove that ρ_k also have a uniformly positive bound from below. By stepsize rule of Algorithm 3.1, we have

$$\begin{aligned}\sigma\|R(x^k,\rho)\| \quad &< \rho\|F(x^k)-F(x^k(\rho))\|\\ &= \rho\|R(x^k,\rho)\|,\end{aligned}$$

i.e., $\rho>\sigma$.

Lemma 3.4 Under Assumption 2.1, and $\|M^+\|\|M\|\leq 1$, we have that the sequence $\{x^k\}$ generated by Algorithm 3.1 is bounded.

Proof. Given $x^* \in X^*$, then we have

$$\begin{aligned}
\|F(x^{k+1}) - F(x^*)\|^2 &= \|P_{\mathcal{K}}[F(x^k) + \varphi\alpha_k d_k] - P_{\mathcal{K}}[F(x^*)]\|^2 \\
&\leq \|F(x^k) - F(x^*) + \varphi\alpha_k d_k\|^2 \\
&= \|F(x^k) - F(x^*)\|^2 + 2\varphi\alpha_k(F(x^k) - F(x^*))^\top d_k + \varphi^2\alpha_k^2\|d_k\|^2 \\
&\leq \|F(x^k) - F(x^*)\|^2 - 2\varphi\alpha_k(1 - \sigma\|NM^+\|)\|R(x^k, \rho_k)\|^2 + \varphi^2\alpha_k^2\|d_k\|^2 \\
&= \|F(x^k) - F(x^*)\|^2 - 2\varphi\alpha_k(1 - \sigma\|NM^+\|)\|R(x^k, \rho_k)\|^2 \\
&\quad + \varphi^2\alpha_k(1-\sigma)\|R(x^k, \rho_k)\|^2 \\
&= \|F(x^k) - F(x^*)\|^2 - \varphi\alpha_k[2(1 - \sigma\|NM^+\|) - \varphi(1-\sigma)]\|R(x^k, \rho_k)\|^2 \\
&\leq \|F(x^k) - F(x^*)\|^2 - \eta\varphi[2(1 - \sigma\|NM^+\|) - \varphi(1-\sigma)]\|R(x^k, \rho_k)\|^2.
\end{aligned} \tag{13}$$

where the third inequality is derived from Lemma 3.2, the third equation uses algorithm 3.1 representation of α_k in Algorithm 3.1. From (13), we obtain

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|M^+(F(x^{k+1}) - p) - M^+(F(x^*) - p)\|^2 \\
&\leq \|M^+\|^2\|F(x^{k+1}) - F(x^*)\|^2 \\
&\leq \|M^+\|^2\|M\|^2\|x^k - x^*\|^2 - \eta\|M^+\|^2\varphi[2(1 - \sigma\|NM^+\|) - \varphi(1-\sigma)]\|R(x^k, \rho_k)\|^2 \\
&\leq \|x^k - x^*\|^2 - \eta\|M^+\|^2\varphi[2(1 - \sigma\|NM^+\|) - \varphi(1-\sigma)]\|R(x^k, \rho_k)\|^2,
\end{aligned} \tag{14}$$

Combining (14) with definition of φ, σ in Algorithm 3.1, we show that the sequence $\|x^k - x^*\|$ is decreasing and nonnegative, it is bounded, and so is also $\{x^k\}$.

Theorem 3.1 Under Assumption 2.1, and $\|M^+\|\|M\| \leq 1$, the sequence $\{x^k\}$ are generated by Algorithm 3.1 converges globally to a solution of GLCP.

Proof. Using (14), we know that the sequence $\|x^k - x^*\|$ is decreasing and nonnegative, it is bounded, it must converges, and we have

$$\sum_{k=0}^{\infty} \|R(x^k, \rho_k)\|^2 \leq \infty,$$

i.e.,

$$\lim_{k\to\infty} \|R(x^k, \rho_k)\| = 0. \tag{15}$$

Thus, we know that any cluster $\bar{x}$ of the sequence $\{x^k\}$ is a solution of GLCP. Since the sequence $\|x^k - x^*\|$ is non-increasing and nonnegative, it is bounded, if we take $x^* = \bar{x}$, then $\{x^k\}$ converges globally to $\bar{x}$.

To establish the global convergence rate of Algorithm 3.1, we will give the following definition.

Definition 3.1. $G(u)$ is said to be $F-$strongly monotone on R^n, if for all $u, v \in R^n$, $\exists \beta > 0$, such that

$$\langle G(u) - G(v), F(u) - F(v)\rangle \geq \beta\|F(u) - F(v)\|^2.$$

Lemma 3.6 Suppose G is $F-$strongly monotone on R^n, for $x^k \in R^n$ and a constant $\rho_k > 0$, we have

$$\frac{\|R(x^k, \rho_k)\|}{2 + \rho_k\|NM^+\|} \leq \|F(x^k) - F(x^*)\| \leq \frac{\rho_k\|NM^+\| + 1}{\rho_k\beta}\|R(x^k, \rho_k)\|. \tag{16}$$

Proof. We assume that x^* be a fixed solution of GLCP, and by Theorem 2.1, we have

$$\langle \rho_k G(x^*), P_{\mathcal{K}}(F(x^k) - \rho_k G(x^k)) - F(x^*)\rangle \geq 0, \tag{17}$$

By Lemma 2.1 (i) and $F(x^*) \in \mathcal{K}$, we know that

$$\langle P_{\mathcal{K}}(F(x^k) - \rho_k G(x^k)) - (F(x^k) - \rho_k G(x^k)), F(x^*) - P_{\mathcal{K}}(F(x^k) - \rho_k G(x^k))\rangle \geq 0, \tag{18}$$

by(17),(18),

$$\langle P_{\mathcal{K}}(F(x^k) - \rho_k G(x^k)) - F(x^k) + \rho_k(G(x^k) - G(x^*)), F(x^*) - P_{\mathcal{K}}(F(x^k) - \rho_k G(x^k))\rangle \geq 0,$$

i.e.,

$$\langle \rho_k(G(x^k) - G(x^*)) - R(x^k, \rho_k), F(x^*) - F(x^k) + R(x^k, \rho_k)\rangle \geq 0,$$

we have

$$\langle \rho_k(G(x^k) - G(x^*)) + (F(x^k) - F(x^*)), R(x^k, \rho_k)\rangle \geq \langle \rho_k(G(x^k) - G(x^*)), F(x^k) - F(u^*)\rangle.$$

i.e.,

$$\langle \rho_k(G(x^k) - G(x^*), F(x^k) - F(x^*)\rangle \leq \|R(x^k, \rho_k)\| \cdot (\rho_k\|G(x^k) - G(x^*)\| + \|F(x^k) - F(x^*)\|).$$

By definition 3.1, there exists $\beta > 0$ such that,

$$\begin{aligned}
\rho_k\beta\|F(x^k) - F(x^*)\|^2 &\leq \langle \rho_k(G(x^k) - G(x^*)), F(x^k) - F(x^*)\rangle \\
&\leq \|R(x^k, \rho_k)\| \cdot (\rho_k\|NM^+\|\|F(x^k) - F(x^*)\| + \|F(x^k) - F(x^*)\|) \\
&= \|R(x^k, \rho_k)\| \cdot (\rho_k\|NM^+\| + 1)\|F(x^k) - F(x^*)\|,
\end{aligned}$$

i.e.,

$$\rho_k\beta\|F(x^k) - F(x^*)\| \leq \|R(x^k, \rho_k)\| \cdot (\rho_k\|NM^+\| + 1), \tag{19}$$

using (19), we know that the right-hand side inequality of (16) holds.

On the other hand, by Lemma 2.2-2.1(ii), we have

$$\begin{aligned}
\|R(x^k, \rho_k)\| = &\quad \|R(x^k, \rho_k) - R(x^*, \rho_k)\| \\
\leq &\quad \|F(x^k) - F(x^*)\| + \|P_{\mathcal{K}}(F(x^k) - \rho_k G(x^k)) - P_{\mathcal{K}}(F(x^*) - \rho_k G(x^*))\| \\
\leq &\quad \|F(x^k) - F(x^*)\| + \|F(x^k) - \rho_k G(x^k) - F(x^*) + \rho_k G(x^*)\| \\
\leq &\quad 2\|F(x^k) - F(x^*)\| + \rho_k\|NM^+\|\|F(x^k) - F(x^*)\| \\
= &\quad (2 + \rho_k\|NM^+\|)\|F(x^k) - F(x^*)\|.
\end{aligned}$$

Thus, the left-hand side inequality of (16) follows.

Theorem 3.2 Suppose G is $F-$strongly monotone on R^n, and $\|M^+\|\|M\| \leq 1$, then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges globally to a solution of GLCP at $R-$linear rate, where $\tau := \eta\varphi[2(1-\sigma\|NM^+\|) - \varphi(1-\sigma)](\sigma\beta/[\|NM^+\|+1])^2 < 1$.

Proof. Using (13), and the right-hand side of (16), we have

$$\begin{aligned}\|F(x^{k+1}) - F(x^*)\|^2 &\leq \|F(x^k) - F(x^*)\|^2 \\ &\quad -\eta\varphi[2(1-\sigma\|NM^+\|) - \varphi(1-\sigma)]\left(\tfrac{\rho_k\beta}{\rho_k\|NM^+\|+1}\right)^2\|F(x^k) - F(x^*)\|^2.\end{aligned}$$

i.e.,

$$\frac{\|F(x^{k+1}) - F(x^*)\|^2}{\|F(x^k) - F(x^*)\|^2} \leq 1 - \eta\varphi[2(1-\sigma\|NM^+\|) - \varphi(1-\sigma)]\left(\frac{\rho_k\beta}{\rho_k\|NM^+\|+1}\right)^2.$$

Combining $\sigma \leq \rho_k \leq 1$, we know that

$$\frac{\|F(x^{k+1}) - F(x^*)\|^2}{\|F(x^k) - F(x^*)\|^2} \leq 1 - \eta\varphi[2(1-\sigma\|NM^+\|) - \varphi(1-\sigma)]\left(\frac{\sigma\beta}{\|NM^+\|+1}\right)^2,$$

i.e.,

$$\begin{aligned}\|F(x^{k+1}) - F(x^*)\| &\leq \sqrt{1-\tau}\|F(x^k) - F(x^*)\| \\ &\leq (\sqrt{1-\tau})^2\|F(x^{k-1}) - F(x^*)\| \\ &\leq \cdots\cdots \\ &\leq (\sqrt{1-\tau})^{k+1}\|F(x^0) - F(x^*)\|.\end{aligned} \tag{20}$$

By $0 < \tau := \eta\varphi[2(1-\sigma\|NM^+\|) - \varphi(1-\sigma)](\sigma\beta/[\|NM^+\|+1])^2 < 1$, then $0 < 1-\tau < 1$. Combining this with (20), we have

$$\|x^{k+1} - x^*\| \leq \|M^+\|\|F(x^{k+1}) - F(x^*)\| \leq \|M^+\|(\sqrt{1-\tau})^{k+1}\|F(x^0) - F(x^*)\|.$$

Thus, the sequence $\{x^k\}$ converges globally to a solution of GLCP at $R-$linear rate.

4 Conclusions

In this paper, we presented a new iterative method for solving GLCP, which ensures that the corrector stepsizes and predictor stepsizes both have a uniformly positive bound from below, under mild conditions, we prove its global convergence. Furthermore, the error bound for GLCP is also given under suitable conditions, based on which we prove that the method has global and R-linear convergence rate.

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