OPTIMAL CONTROL OF LINEAR LOWER ORDER NON-DISPERSIVE WAVES USING STATE VARIABLE SYSTEM APPROACH

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Abstract

We [4] have earlier provided solutions to the problem of the optimal control of lower order waves, using a Hamiltonian approach to solve the model optimization problem with the resulting semi-analytical solutions computationally simulated for analysis. In this paper, a variant of the problem is studied employing a sequence of constraint differential equations in the state functions, arising from the Hamiltonian approach. As usual, our results are computationally simulated for analysis, comparing results with those earlier obtained by Reju et al [4]

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1. Introduction

As stated in [4], the higher order non-dispersive or hyperbolic wave equation with prototype:

$$\frac{\partial^2 z(x,t)}{\partial t^2} = c_0^2 \frac{\partial^2 z(x,t)}{\partial x^2} \tag{1.1}$$

arises in many fields such as acoustics, elasticity and electromagnetism. The lower order form:

$$\frac{\partial z(x,t)}{\partial t} + c_0 \frac{\partial z(x,t)}{\partial x} = 0 \tag{1.2}$$

is a simpler variant of (1.1).

We consider the following constrained optimal control problem for a quadratic objective functional:

Problem 1:

Minimize J[z(x, t), u(x, t)] = Minimize $\int_0^1\int_0^1 [z^2(x,t) + u^2(x,t)]dxdt$ (1.3)

subject to the lower order wave propagation:

$$\frac{\partial z(x,t)}{\partial t} + c_0 \frac{\partial z(x,t)}{\partial x} = u(x,t) \tag{1.4}$$

with the following boundary and initial conditions:

$$\begin{aligned} z(0,t) &= z(1,t) = 0 \\ z(x,0) &= z_0(x) \end{aligned} \tag{1.5}$$

where u(x, t) is our control function and c_0 is a constant, being the speed of propagation.

Now, formulating a Hamiltonian akin to that of Singh and Titli [5], we have:

$$H(z(x,t),u(x,t)) = f(z(x,t),u(x,t)) + \lambda^T [g(z(x,t),u(x,t))] \tag{1.6}$$

where

$$f(z(x,t),u(x,t)) = z^2(x,t) + u^2(x,t)$$
$$g(z(x,t),u(x,t)) = -c_0 \frac{\partial z(x,t)}{\partial x} + u(x,t) \qquad (1.7)$$

The optimality conditions are as follows:

$$\frac{\partial H}{\partial z} = -2z = 0$$
$$\frac{\partial H}{\partial u} = 2u + \lambda = 0 \qquad (1.8)$$
$$\frac{\partial H}{\partial \lambda} = -c_0 \frac{\partial z}{\partial x} + u = \frac{\partial z}{\partial t}$$

From the above equations, we have:

$$\lambda = -2u \ \text{ and } \ z = -\frac{\partial u}{\partial t} \qquad (1.9)$$

Equation (1.9) describes the wave propagation displacement as a time derivative of the control function akin to Reju's result [1].

We shall now proceed to the focus of the paper in the next section.

2 Model Problem

We now assume Fourier solutions of the following forms:

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t)\sin(\pi i x)$$
$$u(x,t) = \sum_{i=1}^{\infty} u_i(t)\sin(\pi i x) \qquad (2.1)$$

In [4], the following problem was considered

$$\text{Minimize } \int_0^1 [u_{it}^2(t) + u_i^2(t)]dt$$

subject to a system of equations:

$$\frac{\partial^2 u_1}{\partial t^2} + c_0\pi.1.\cot(\pi.1.x)\frac{\partial u_1}{\partial t}u_1 + u_1 = 0$$
$$\frac{\partial^2 u_2}{\partial t^2} + c_0 2\pi\cot(2\pi x)\frac{\partial u_2}{\partial t}u_2 + u_2 = 0$$
$$\dots$$
$$\frac{\partial^2 u_n}{\partial t^2} + c_0 n\pi\cot(n\pi x)\frac{\partial u_n}{\partial t}u_n + u_n = 0$$

However, in this paper, we substitute (2.1) in our problem to convert Problem 1 to the following problem:

Problem 2:

$$\text{Minimize } \int_0^1 [u_{it}^2(t) + u_i^2(t)]dt \quad (2.2)$$

subject to a system of state variable equations:

$$\begin{aligned}
&\frac{\partial z_1}{\partial t} + c_0\pi.1.\cot(\pi.1.x)z_1 = u_1\\
&\frac{\partial z_2}{\partial t} + c_0 2\pi\cot(2\pi x)z_2 = u_2\\
&\dots\\
&\frac{\partial z_n}{\partial t} + c_0 n\pi\cot(n\pi x)z_n = u_n
\end{aligned} \quad (2.3)$$

Using (1.9) and (2.1), we solve the general equation of (2.3) to obtain the control and state functions given by the following:

$$u(x,t) = \sum_{i=1}^{\infty}\left(\frac{u_0 t^2}{2} + \frac{z_0\exp(-kt)}{k} + u_0 + \frac{z_0\tan(\pi ix)}{\pi ic_0}\right)\sin(\pi ix) \quad (2.4)$$

where $k = -c_0 i\pi\cot(\pi ix)$

and z(x, t) is given by:

$$z(x,t) = \sum\left(z_0\exp((-c_0 i\pi\cot(\pi ix))t + u_0 t\right)\sin(\pi ix) \quad (2.5)$$

3. Numerical Simulations of Results and Analysis

Below, we present the surface plots for the control function U = u(x, t) and the state function Z = z(x, t):

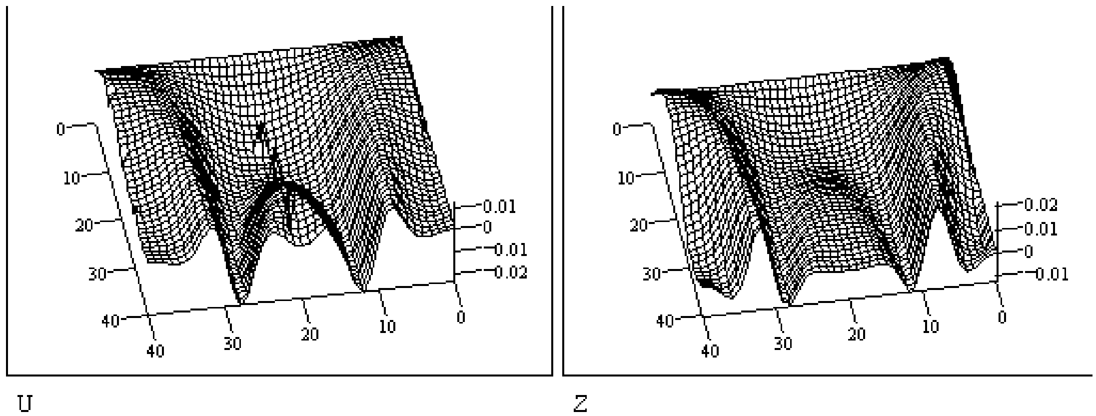

Figure 3.1: Surface Plot with Periodic Cosine Iterates and Small Space Dimension

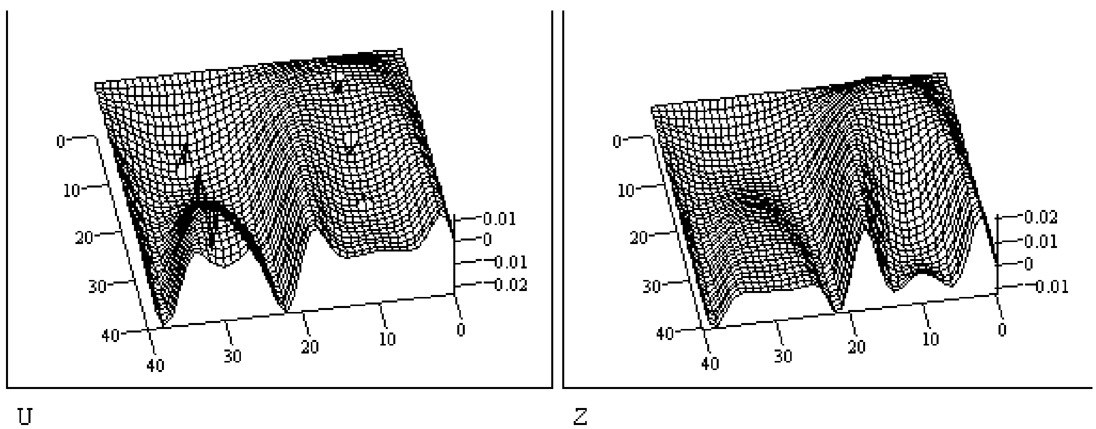

Figure 3.2: Surface Plot with Periodic Sine Iterates and Small Space Dimension

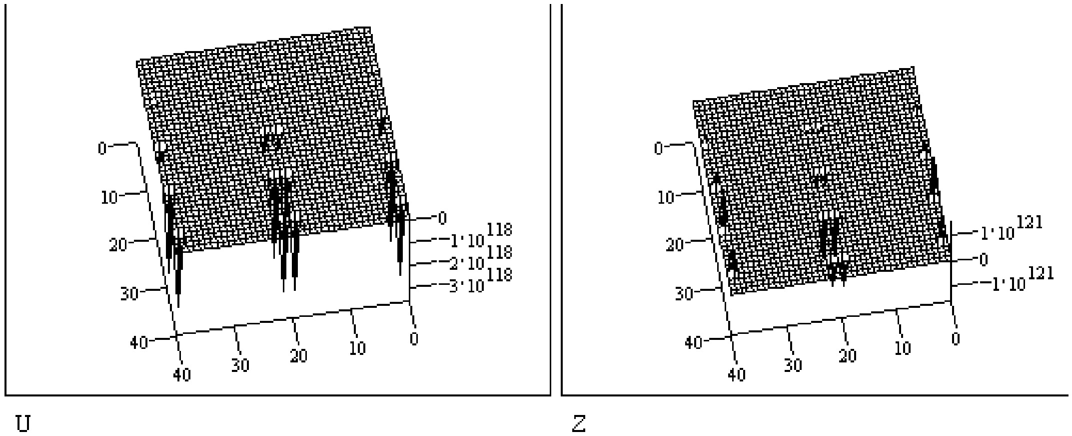

Figure 3.3: Surface Plot with Cosine Periodic Iterates, Small Initial Control and Increased Space Dimension

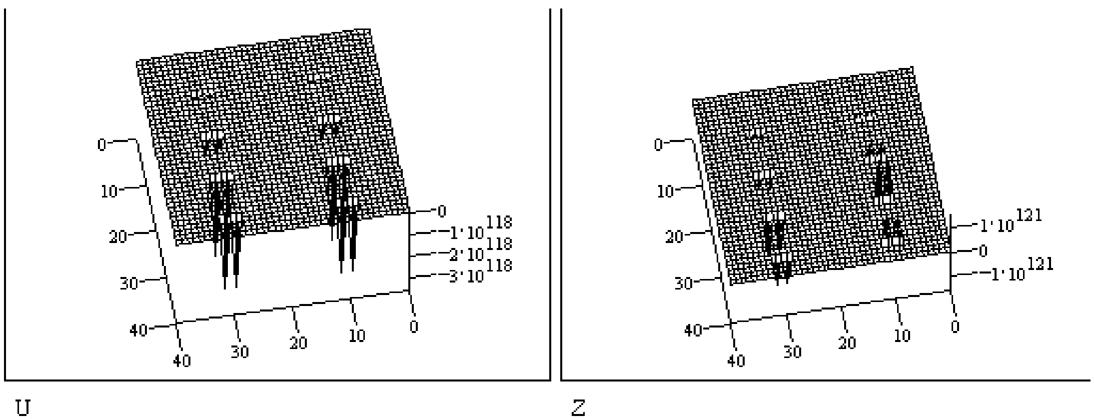

Figure 3.4: Surface Plot with Sine Periodic Iterates, Small Initial Control and Increased Space Dimension

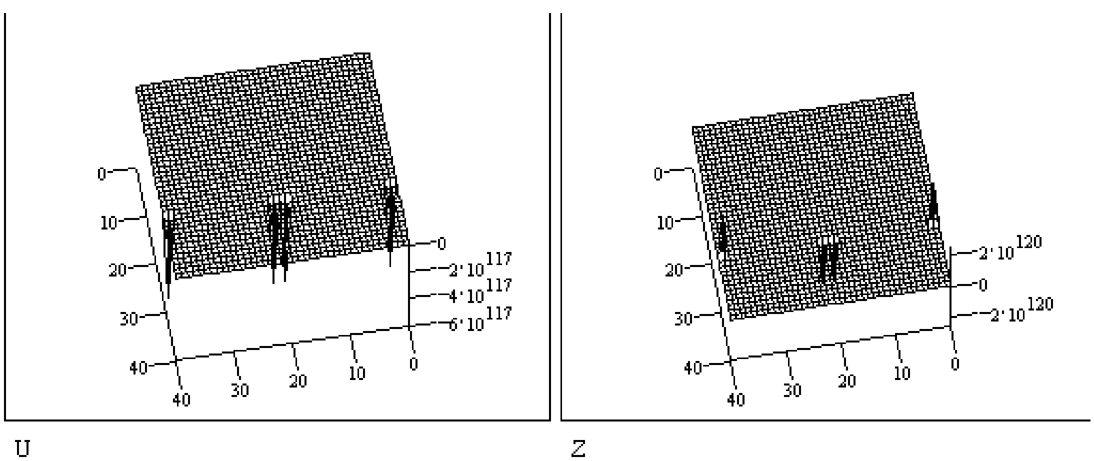

Figure 3.5: Surface Plot Using Fig 3.3 Data with Reduced Space Dimension

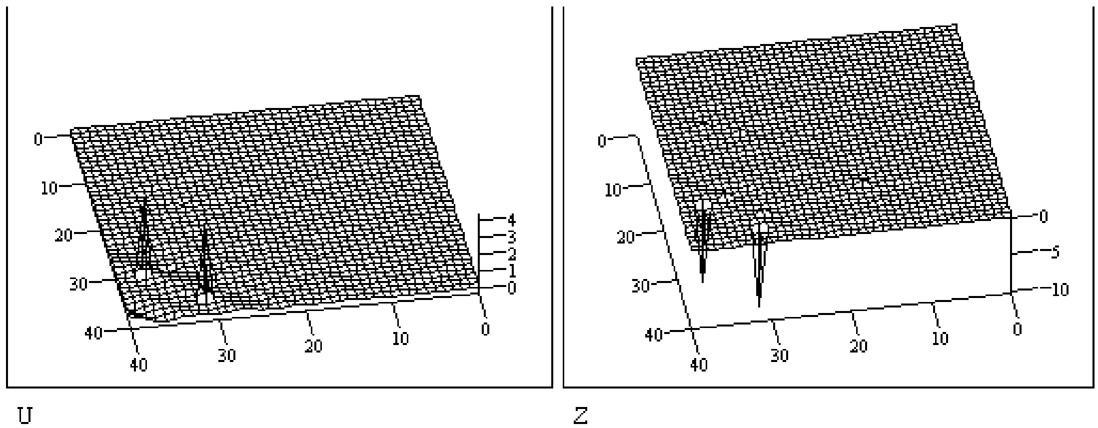

Figure 3.6: Surface Plot Using Fig 3.1 Data with Non Periodic Iterate Definition

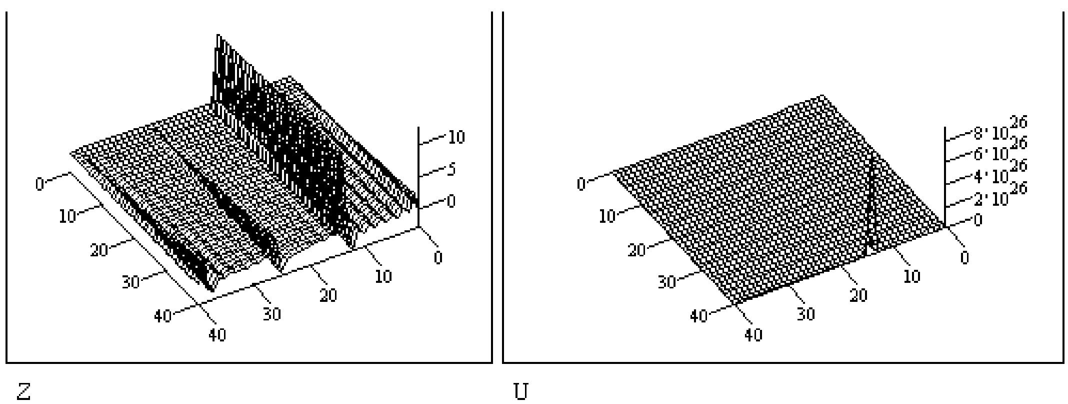

Figure 3.7: Surface Plot with Linear Iterate Definition and Small Space Dimension

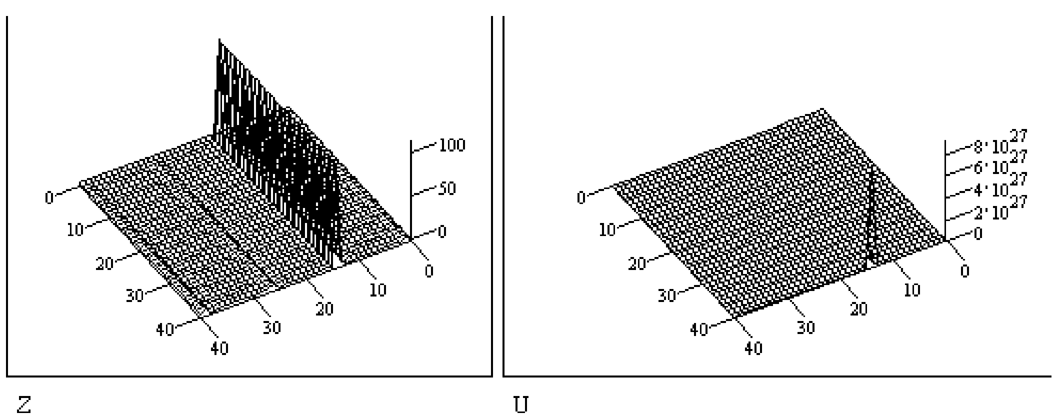

Figure 3.8: Surface Plot with Linear Iterate Definition and Increased Initial State

Closely studying the above simulated solutions, the following remarks are in order:

i. The optimal values depend very much on the space dimension either for periodic or non-periodic definition of the iterates.
ii. As we have in this paper attempted to introduce periodicity into the iterate's definition to see how this affects the optimal values, the results obviously reveal that periodicity increases the optimum values as shown in Figure 3.1 when compared with Figure 3.6.
iii. Figure 3.6 simply depicts the profiles when the periodicity introduced by the transcendental function is removed. As shown, this gives better disturbance-free (or stable) region.
iv. With linear iterate definition akin to those employed throughout our results' simulations in [4], Figures 3.4 and 3.5 of this paper reveal that with increased initial state value, higher optimal control values are obtained alongside with increased optimal state, however, characterised by reduced multi-modal optimum propagations of the state. Moreover, the wave propagation profile depicts a situation where energy released at some other points seems to have been absorbed by the dominating point of disturbance to increase its optimum release.

4 Conclusion

The model variant of our earlier work [4] presented in this paper seems comparatively simpler but yet with some good simulation similarities as seen in Figures 3.4 and 3.5 of this paper and the last two figures of [4]. The wave propagation profiles for the cited figures are both of dominantly stable region with optimum points of the control occurring at the boundary, within the same vicinity. This for example underscores the importance of the state variable approach employed in this paper, for the first time in the series of research models earlier studied by Reju [1, 2].

The introduction of periodicity in the definition of the iterates has its significance as seen in the simulated results, since the whole results in this paper reveal that the way iterates are defined can significantly affect our computationally simulated results.

On the whole, our results show, as expected, great dependence of the optimum values on the initial values and the space dimensions. Further studies of the nonlinear models of the lower order wave propagations should take into consideration some of the vital dimensions introduced in the paper so as to provide a good direction towards a unified approach.

References

1. Reju, S. A. (1995) Computational Optimization in Mathematical Physics, Ph.D. Thesis, University of Ilorin, Ilorin, Nigeria.

2. Reju, S.A., Ibiejugba, M.A. and Evans, D.J. (2001) Optimal Control of the Wave Propagation Problem with the Extended Conjugate Gradient Method, Inter. J. Comp. Math., Vol. 77, Number 3, pp 425-439.

3. Reju, C. O. (2003) Optimal Control of Quasi-Linear Non-Dispersive Waves, M.Tech Thesis, Federal University of Technology, Minna, Nigeria.

4. Reju, S. A., Reju, C. O. and Akinwande, N. I. (2012) Optimal Control of Linear Lower Order Non-Dispersive Waves – submitted for publication.

5. Singh, M. G. and Titli, A. (1978) Systems: Decomposition, Optimization and Control, Pergamon Press.

6. Waziri, V. O. and Reju, S. A. (2005) The Control Operator for the One-Dimensional Energized Wave Equation; AU. J. T. 9(4), 243-247.

7. Waziri, V. O. (2006) Optimal Control of Energized Wave Equations Using Extended Conjugate Gradient Method, Ph.D. Thesis, Federal University of Technology, Minna, Nigeria.

8. Waziri, V. O. and Reju, S. A. (2006) The Analytical Solutions of the One-Dimensional Energy, AU. J. T. 10(2), 124-128.

9. Waziri, V. O. and Reju, S. A. (2006) Control Operator for the Two-Dimensional Energized Wave Equation, Leonardo J. Sc, (9), Pp33 -44, ISSN 1583-0233,

10. Waziri, V. O. and Reju, S. A. (2006) Analysis of the Two-dimensional Diffusion Equation With a Source, LEJPT, (9), Pp43-54, ISSN 1583-1078

11. Whitham, G. B., (1974) Linear and Nonlinear Waves, John Wiley & Sons, Inc.