OPTIMAL CONTROL OF LINEAR LOWER ORDER NON-DISPERSIVE WAVES

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Abstract

We can majorly distinguish two main classes of waves, namely, dispersive and non-dispersive waves. The latter class, also termed the hyperbolic waves, are so-called since they can be formulated in terms of hyperbolic partial differential equations. Generally, there are waves that exhibit both types of behaviours but dispersive waves are not classified as easily as non-dispersive. This paper considers the optimal control of lower order wave which has its essential role in applications. For example, the higher order waves, whose optimal control was studied by Reju [1, 2], often carry the "first signal" when combined with lower order waves, but the main disturbance or propagation travels with the lower order waves as confirmed by some of our simulated results in this work when compared with the higher order control problem of Reju [1, 2]. A Hamiltonian approach is employed to solve the model optimization problem with the resulting semi-analytical solutions computationally simulated for analysis.

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1. Introduction

Avoiding any restrictive definition, a wave has been defined as any recognizable signal that is transferred from one part of a medium to another with a recognizable velocity of propagation while the signal may be any disturbance feature such as a maximum (an optimum) or an abrupt change in some quantity, provided it can be recognised and its location at any time can be determined [5].

The higher order non-dispersive or hyperbolic wave equation with prototype:

$$\frac{\partial^2 z(x,t)}{\partial t^2} = c_0^2 \frac{\partial^2 z(x,t)}{\partial x^2} \tag{1.1}$$

arises in many fields such as acoustics, elasticity and electromagnetism [5]. The lower order form:

$$\frac{\partial z(x,t)}{\partial t} + c_0 \frac{\partial z(x,t)}{\partial x} = 0 \tag{1.2}$$

is a simpler variant of (1.1), possibly the simplest, but not without its importance in applications.

Most main wave propagations travel with the lower order waves. Many wave motions have been studied which lead to equation of the form (1.2). Some examples are flood waves, waves in glaciers, waves in traffic flow, and certain wave phenomena in chemical reactions. The study of (1.2) is therefore in order from its immense applications. However, it should also be noted that (1.2) is also obtainable from a factoring of (1.1) into two waves.

2 Optimal Control Model Problem

We consider the following constrained optimal control problem for a quadratic objective functional:

Problem 1

$$\text{Minimize } J[z(x, t), u(x, t)] = \text{Minimize } \int_0^1\int_0^1 [z^2(x,t) + u^2(x,t)]dxdt \tag{2.1}$$

subject to the lower order wave propagation:

$$\frac{\partial z(x,t)}{\partial t} + c_0 \frac{\partial z(x,t)}{\partial x} = u(x,t) \tag{2.2}$$

with the following boundary and initial conditions:

$$\begin{aligned} z(0,t) &= z(1,t) = 0 \\ z(x,0) &= z_0(x) \end{aligned} \tag{2.3}$$

where u(x, t) is our control function and c_0 is a constant, being the speed of propagation.

We formulate a Hamiltonian akin to that of Singh and Titli [4] as follows:

$$H(z(x,t),u(x,t)) = f(z(x,t),u(x,t)) + \lambda^T[g(z(x,t),u(x,t))] \tag{2.4}$$

where

$$\begin{aligned} f(z(x,t),u(x,t)) &= z^2(x,t) + u^2(x,t) \\ g(z(x,t),u(x,t)) &= -c_0 \frac{\partial z(x,t)}{\partial x} + u(x,t) \end{aligned} \tag{2.5}$$

The optimality conditions are as follows:

$$\begin{aligned} \frac{\partial H}{\partial z} &= -2z = 0 \\ \frac{\partial H}{\partial u} &= 2u + \lambda = 0 \\ \frac{\partial H}{\partial \lambda} &= -c_0 \frac{\partial z}{\partial x} + u = \frac{\partial z}{\partial t} \end{aligned} \tag{2.6}$$

From the above equations, we have:

$$\lambda = -2u \text{ and } z = -\frac{\partial u}{\partial t} \tag{2.7}$$

Equation (2.7) describes the wave propagation displacement as a time derivative of the control function akin to Reju's result [2]. We now assume Fourier solutions of the following forms:

$$\begin{aligned} z(x,t) &= \sum_{i=1}^{\infty} z_i(t)\sin(\pi i x) \\ u(x,t) &= \sum_{i=1}^{\infty} u_i(t)\sin(\pi i x) \end{aligned} \tag{2.8}$$

Substituting the above in our problem, we now have Problem 1 becoming the following:

Problem 2

$$\text{Minimize } \int_0^1 [u_{it}^2(t) + u_i^2(t)]dt \tag{2.9}$$

subject to a system of equations:

$$\begin{aligned}
&\frac{\partial^2 u_1}{\partial t^2} + c_0 \pi.1.\cot(\pi.1.x)\frac{\partial u_1}{\partial t}u_1 + u_1 = 0 \\
&\frac{\partial^2 u_2}{\partial t^2} + c_0 2\pi\cot(2\pi x)\frac{\partial u_2}{\partial t}u_2 + u_2 = 0 \\
&\qquad\qquad \ldots \\
&\frac{\partial^2 u_n}{\partial t^2} + c_0 n\pi\cot(n\pi x)\frac{\partial u_n}{\partial t}u_n + u_n = 0
\end{aligned} \tag{2.10}$$

Solving the general equation of (2.10), we have our state and control functions given by the following:

$$\begin{aligned}
u(x,t) &= \left(\frac{u_t(0,x) - \lambda_2 u_0(x)}{\lambda_1 - \lambda_2}\right)\exp(\lambda_1 t) + \left(u_0(x) - \frac{u_t(0,x) - \lambda_2 u_0(x)}{\lambda_1 - \lambda_2}\right)\exp(\lambda_2 t) \\
z(x,t) &= \lambda_1\left(\frac{u_t(0,x) - \lambda_2 u_0(x)}{\lambda_1 - \lambda_2}\right)\exp(\lambda_1 t) + \lambda_2\left(u_0(x) - \frac{u_t(0,x) - \lambda_2 u_0(x)}{\lambda_1 - \lambda_2}\right)\exp(\lambda_2 t)
\end{aligned} \tag{2.11}$$

where λ_1 and λ_2 are given by:

$$\begin{aligned}
\lambda_1 &= -c_0 n\pi\cot(n\pi x) + \frac{\sqrt{c_0^2 n^2 \pi^2 \cot^2(n\pi x) - 4}}{2} \\
\lambda_2 &= -c_0 n\pi\cot(n\pi x) - \frac{\sqrt{c_0^2 n^2 \pi^2 \cot^2(n\pi x) - 4}}{2}
\end{aligned} \tag{2.12}$$

Explicitly expressed, (2.11) equations are given by:

$$u(x,t)=\left(\frac{u_t(0,x)-[-c_0n\pi\cot(n\pi x)+\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]u_0(x)}{[-c_0n\pi\cot(n\pi x)+\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]-[-c_0n\pi\cot(n\pi x)-\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]}\right)$$

$$.\exp\left([-c_0n\pi\cot(n\pi x)+\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]t\right)$$

$$+\left(u_0(x)-\frac{u_t(0,x)-[-c_0n\pi\cot(n\pi x)-\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]u_0(x)}{[-c_0n\pi\cot(n\pi x)+\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]-[-c_0n\pi\cot(n\pi x)-\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]}\right)$$

$$.\exp\left([-c_0n\pi\cot(n\pi x)-\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]t\right)$$

(2.13)

and

$$z(x,t)=[-c_0n\pi\cot(n\pi x)+\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]$$

$$.\left(\frac{u_t(0,x)-[-c_0n\pi\cot(n\pi x)+\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]u_0(x)}{[-c_0n\pi\cot(n\pi x)+\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]-[-c_0n\pi\cot(n\pi x)-\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]}\right)$$

$$.\exp\left([-c_0n\pi\cot(n\pi x)+\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]t\right)+[-c_0n\pi\cot(n\pi x)-\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]$$

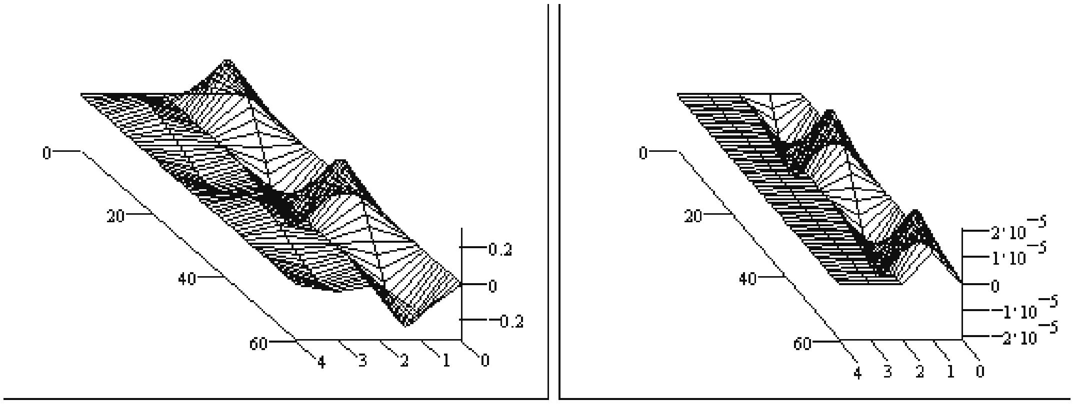
$$.\left(u_0(x)-\frac{u_t(0,x)-[-c_0n\pi\cot(n\pi x)-\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]u_0(x)}{[-c_0n\pi\cot(n\pi x)+\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]-[-c_0n\pi\cot(n\pi x)-\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]}\right)$$

$$.\exp\left([-c_0n\pi\cot(n\pi x)-\frac{\sqrt{c_0^2n^2\pi^2\cot^2(n\pi x)-4}}{2}]t\right)$$

(2.14)

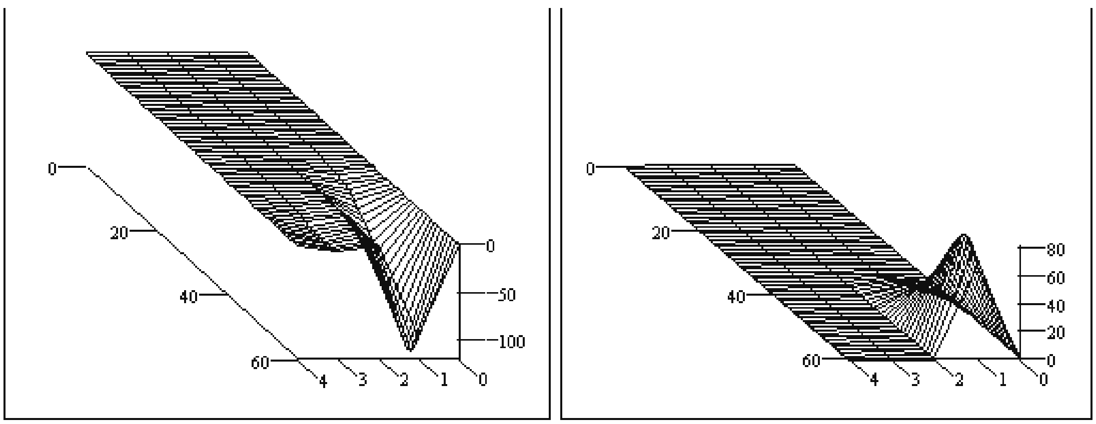
3. Numerical Simulations of Results and Analysis

Below, we present the surface plots for the state and control functions:



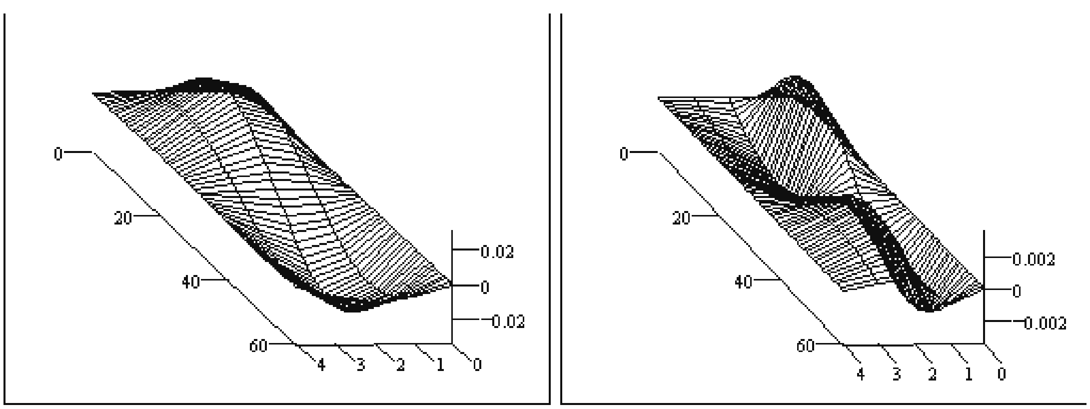
u(x, t) z(x, t)

Figure 3.1: Optimal Control and State for Small Wave Velocity



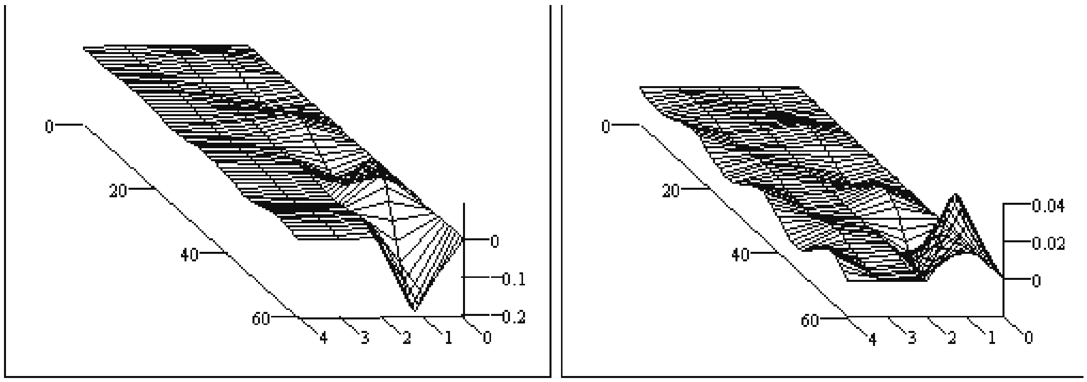
u(x, t) z(x, t)

Figure 3.2: Optimal Control and State with Small Initial Input Velocity



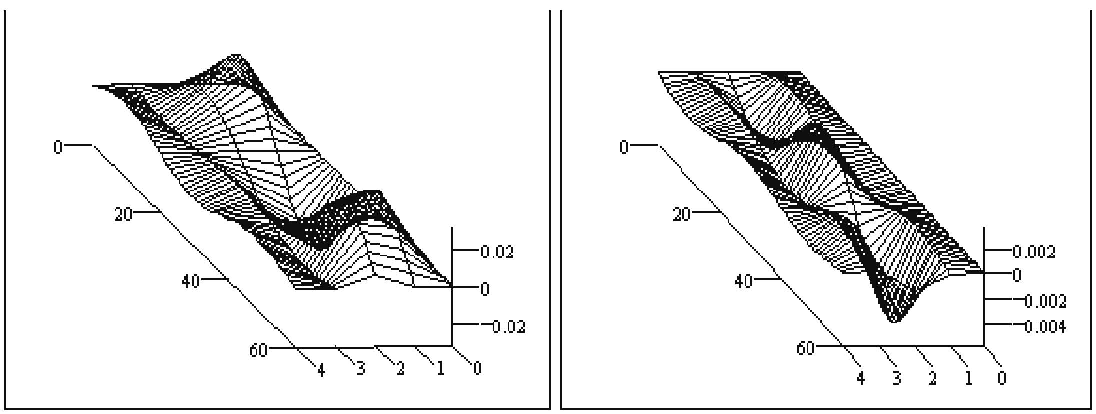
u(x, t) z(x, t)

Figure 3.3: Optimal Control and State with Smaller Initial Input (Profile Similar to Reju [2])



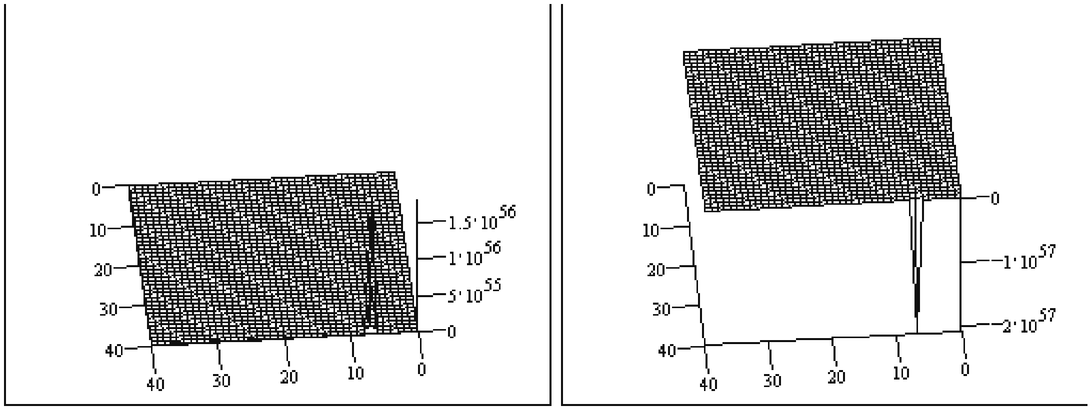
1 u(x, t) z(x, t)

Figure 3.4: Optimal Control and State with Increased Number of Iterates (Using 3.3 Parameters)



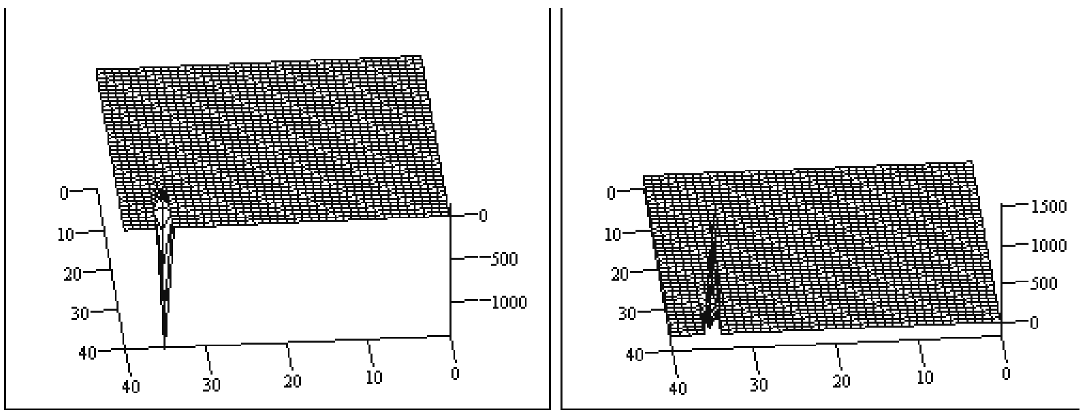
I u(x, t) Z z(x, t)

Figure 3.5: Optimal Control and State with further Increased Number of Iterates (Using 3.3 Parameters)



u(x, t) z(x, t)

Figure 3.6: Optimal Control and State with increased Space Dimension (Using 3.4 Parameters)



u(x, t) z(x, t)

Figure 3.7: Optimal Control and State with Reduced Space Dimension (Using 3.4 Parameters)

Studying the above simulated solutions for the propagation phenomena, our solutions reveal restricted space disturbances until when the space dimension is compensated with increased number of mesh elements. The situation also leads to good stability phenomena accompanied with better optima both in the state and control functions.

A very notable result arising from the above simulated solutions are some propagation profiles obtained which are very similar to those obtained by Reju [2] for higher order waves. This simply confirms the claim that most main disturbances for higher order waves do travel with lower other propagations as stated in our introduction.

The seeming abrupt phenomena observed in Figures 3.6 and 3.7 are characteristics of some physical phenomena where a seeming stable large region suddenly experiences an abrupt disturbance within a small sub-region. Moreover, it should be noted that the optimum values dominantly occur at the boundaries as known for most models and physical situations. Figure 3.2, for example depicts the subduction zone phenomenon in plate tectonics or earthquake studies as studied by Reju [1].

4 Conclusion

Evidently, the results in this paper have shown the great relevance of studying lower order wave propagations. The confirmation that most higher order wave disturbances travel with lower order waves obtained from our results is quite a merit. Moreover, the results suggest the potential of lower order waves as tools for modelling some physical propagation problems arising in dynamics. Nonlinear cases, though more mathematically demanding, are being presently studied by the second author.

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