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Some developments in general mixed quasi variational inequalities

Abdellah Bnouhachem[a,b1],**Mohamed Khalfaoui**[c] and **Hafida Benazza**[c]

[a]School of Management Science and Engineering, Nanjing University, Nanjing, 210093, P.R. China.

[b]Ibn Zohr University, ENSA, BP 32/S, Agadir, Morocco.

[c]Ecole Supérieure de Technologie de Salé, Mohamed 5 University, Agdal-Rabat, Morocco.

Abstract. In this paper, we use the resolvent operator to suggest and analyze two new numerical methods for solving general mixed quasi variational inequalities coupled with new directions and new step sizes. Under certain conditions, the global convergence of the both methods is proved. Our results can be viewed as significant extensions of the previously known results for general mixed quasi variational inequalities.

Key word. General mixed quasi variational inequalities, self-adaptive rules, pseudomonotone operators, resolvent operator.

1 Introduction

Variational inequality has become a rich of inspiration in pure and applied mathematics. In recent years, classical variational inequality problems have been extended and generalized to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences, see [1-39] and the references therein. The projection and contraction method and its invariant forms represent an important tool for finding the approximation solution of various types of variational inequalities and complementarity problems. In recent years variational inequalities have been extended in various directions

[1]Corresponding author. E-mail: babedallah@yahoo.com.

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using novel and innovative techniques. A useful and important generation of variational inequalities is the general mixed variational inequality containing a nonlinear term φ. Due to the presence of the nonlinear term the projection method and its variant forms can not be applied to suggest iterative algorithms for solving mixed variational inequalities. To overcome these drawbacks, some iterative methods have been suggested for solving the general mixed quasi variational inequalities. For example, if the bifunction is proper, convex and lower semicontinuous function with respect to the first argument, then Noor [28, 29] has shown that the general mixed quasi variational inequalities are equivalent to the fixed-point problems and the implicit resolvent equations using the resolvent operator technique. This equivalent formulation has been used to suggest and analyze some iterative methods. It has been proved that the convergence of these methods requires that the operator is both strongly monotone and Lipschitz continuous. Secondly, it is very difficult to evaluate the resolvent of the operator except for very simple cases. To overcome this disadvantage, Noor and Noor [33] employed some alternative equivalent formulations to suggest and analyze modified resolvent iterative method for general mixed quasi variational inequalities, where the skew-symmetry of the nonlinear bifunction plays a crucial part in the convergence analysis of these methods. Inspired and motivated by on going research in this direction, we suggest and consider two iterative methods for solving the general mixed quasi variational inequalities involving the nonlinear term, which is the main motivation of this paper. We prove the global convergence of these new methods under some mild and suitable conditions. Since the general mixed quasi variational inequalities includes the general variational inequalities, quasi variational inequalities and complementarity problems as special cases, results obtained in this paper continue to hold for these problems. It is expected that these results may inspire and motivate others to find novel and innovative applications in various branches of pure and applied sciences.

2 Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let K be a closed convex set in H and $T, g : H \to H$ be two operators. Let $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ be a continuous bifunction. We consider the problem of finding

$u^* \in H$ such that

$$\langle T(u^*), g(v) - g(u^*)\rangle + \varphi(g(v), g(u^*)) - \varphi(g(u^*), g(u^*)) \geq 0, \quad \forall v \in H. \tag{2.1}$$

is called the general mixed quasi variational inequality, see Noor and Noor [33]. We note that, if the bifunction $\varphi(.,.)$ is a proper, convex and lower semicontinuous function with respect to the first argument, then problem (2.1) is equivalent to finding $u \in H$ such that

$$0 \in T(u) + \partial\varphi(g(u), g(u)), \tag{2.2}$$

which is known as finding the zero of the sum of monotone operators. See also [26, 27] for applications and numerical methods of problem (2.2).

For $\varphi(v, u^*) = \varphi(v), \forall u^* \in H$, problem (2.1) reduces to finding $u^* \in H$ such that

$$\langle T(u^*), g(v) - g(u^*)\rangle + \varphi(g(v)) - \varphi(g(u^*)) \geq 0, \quad \forall v \in H, \tag{2.3}$$

which is known as the general mixed variational inequality, see Noor [28].

If $\varphi(.,.) = \varphi(.)$ is an indicator function of a closed convex set K in H, then the problem (2.1) is equivalent to finding $u^* \in H$ such that $g(u^*) \in K$ and

$$\langle T(u^*), g(u) - g(u^*)\rangle \geq 0, \qquad \forall g(u) \in K. \tag{2.4}$$

Problem (2.4) is called the general variational inequality, which was first introduced and studied by Noor [22] in 1988. For the applications, formulation and numerical methods of general variational inequalities (2.4), we refer the reader to the survey [4, 6, 17, 31].

If $g \equiv I$, then the problem (2.4) is equivalent to finding $u^* \in K$ such that

$$\langle T(u^*), v - u^*\rangle \geq 0, \qquad \forall v \in K, \tag{2.5}$$

is called as the classical variational inequality problem , which is was introduced by Stampacchia [38] in 1964. For the recent applications, numerical techniques and physical formulation, see [1-39].

We also need the following well known results and concepts.

Definition 2.1 $\forall u, v \in H$, the operator $T : H \longrightarrow H$ is said to be g-pseudomonotone, if

$$\langle T(u), g(v) - g(u)\rangle \geq 0 \quad \text{implies} \quad \langle T(v), g(v) - g(u)\rangle \geq 0.$$

Definition 2.2 The bifunction $\varphi(.,.)$ is said to be skew-symmetric, if,

$$\varphi(u,u)-\varphi(u,v)-\varphi(v,u)+\varphi(v,v)\geq 0, \quad \forall u,v\in H. \tag{2.6}$$

Clearly, if the bifunction $\varphi(.,.)$ is linear in both arguments, then,

$$\varphi(u,u)-\varphi(u,v)-\varphi(v,u)+\varphi(v,v)=\varphi(u-v,u-v)\geq 0, \quad \forall u,v\in H,$$

which shows that the bifunction $\varphi(.,.)$ is nonnegative.

Definition 2.3 [11] For any maximal operator T, the resolvent operator associated with T, for any $\rho>0$, is defined as

$$J_T(u)=(I+\rho T)^{-1}(u), \qquad \forall u\in H. \tag{2.7}$$

Remark 2.1 It is well known that the subdifferential $\partial\varphi(.,.)$ of a convex, proper and lower-semicontinuous function $\varphi(.,.):H\times H\longrightarrow R\cup\{+\infty\}$ is a maximal monotone with respect to the first argument, we can define its resolvent by

$$J_{\varphi(u)}=(I+\rho\partial\varphi(.,u))^{-1}\equiv(I+\rho\partial\varphi(u))^{-1}, \tag{2.8}$$

where $\partial\varphi(u)\equiv\partial\varphi(.,u)$.

The resolvent operator $J_{\varphi(u)}$defined by (2.8) has the following characterization,

Lemma 2.1 [29] For a given $z\in H$, $u\in H$ satisfies the inequality

$$\langle u-z,v-u\rangle+\rho\varphi(v,u)-\rho\varphi(u,u)\geq 0, \quad \forall v\in H, \tag{2.9}$$

if and only if

$$u=J_{\varphi(u)}[z],$$

where $J_{\varphi(u)}$ is resolvent operator defined by (2.8).

It follows from Lemma 2.1 that

$$\langle J_{\varphi(u)}[w]-w,v-J_{\varphi(u)}[w]\rangle+\rho\varphi(v,J_{\varphi(u)}[w])-\rho\varphi(J_{\varphi(u)}[w],J_{\varphi(u)}[w])\geq 0, \qquad \forall u,v,w\in H \tag{2.10}$$

The following result can be proved by using Lemma 2.1.

Lemma 2.2 [24] $u^*\in H$ is solution of problem (2.1) if and only if $u^*\in H$ satisfies the relation:

$$g(u^*)=J_{\varphi(u^*)}[g(u^*)-\rho T(u^*)], \tag{2.11}$$

where $\rho>0$.

From Lemma 2.2, it is clear that u is solution of (2.1) if and only if u is a zero point of the function

$$r(u,\rho) := g(u) - J_{\varphi(u)}[g(u) - \rho T(u)].$$

Lemma 2.3 [4] For all $u \in H$ and $\rho' \geq \rho > 0$, it holds that

$$\|r(u,\rho')\| \geq \|r(u,\rho)\| \tag{2.12}$$

and

$$\frac{\|r(u,\rho')\|}{\rho'} \leq \frac{\|r(u,\rho)\|}{\rho}. \tag{2.13}$$

Lemma 2.4 For all $v, w \in H$, we have

$$\|J_{\varphi(u)}(w) - J_{\varphi(u)}(v)\|^2 \leq \langle w - v, J_{\varphi(u)}(w) - J_{\varphi(u)}(v)\rangle. \tag{2.14}$$

Proof. By using (2.10), we get

$$\langle w - J_{\varphi(u)}(w), J_{\varphi(u)}(w) - J_{\varphi(u)}(v)\rangle + \rho\varphi(J_{\varphi(u)}(v), J_{\varphi(u)}(w)) - \rho\varphi(J_{\varphi(u)}(w), J_{\varphi(u)}(w)) \geq 0 \tag{2.15}$$

and

$$\langle v - J_{\varphi(u)}(v), J_{\varphi(u)}(v) - J_{\varphi(u)}(w)\rangle + \rho\varphi(J_{\varphi(u)}(w), J_{\varphi(u)}(v)) - \rho\varphi(J_{\varphi(u)}(v), J_{\varphi(u)}(v)) \geq 0. \tag{2.16}$$

Adding (2.15) and (2.16), and using the skew-symmetry of the bifunction $\varphi(.,.)$, we obtain

$$\langle v - w, J_{\varphi(u)}(v) - J_{\varphi(u)}(w)\rangle \geq \|J_{\varphi(u)}(v) - J_{\varphi(u)}(w)\|^2.$$

□

Throughout this paper, we make following assumptions.

Assumptions:

- H is a finite dimension space.
- g is homeomorphism on H i.e., g is bijective, continuous and g^{-1} is continuous.
- T is continuous and g-pseudomonotone operator on H.
- The bifunction $\varphi(.,.)$ is skew-symmetric.
- The solution set of problem (2.1) denoted by S^*, is nonempty.

3 Main results

In this section, we suggest and analyze the new resolvent methods for solving general mixed quasi variational inequalities (2.1). For given $u^k \in H$ and $\rho_k > 0$, each iteration of the first method consists of three steps, the first step offers $g(\tilde{u}^k)$, the second step makes $g(\bar{u}^k)$ and the third step produces the new iterate $g(u^{k+1})$.

Algorithm 3.1

Step 1. Given $u^0 \in H$, $\epsilon > 0$, $\rho_0 = 1$, $\nu > 1$, $\mu \in (0, \sqrt{2})$, $\tau \in (0,1)$, $\eta_1 \in (0,\tau)$, $\eta_2 \in (\tau, \nu)$ and let $k = 0$.

Step 2. If $\|r(u^k, 1)\| \leq \epsilon$, then stop. Otherwise, go to Step 3.

Step 3. 1) For a given $u^k \in H$, calculate the two predictors

$$g(\tilde{u}^k) = J_{\varphi(u^k)}[g(u^k) - \rho_k T(u^k)], \tag{3.1a}$$

$$g(\bar{u}^k) = J_{\varphi(u^k)}[g(\tilde{u}^k) - \rho_k T(\tilde{u}^k)]. \tag{3.1b}$$

2) If $\|r(\tilde{u}^k, 1)\| \leq \epsilon$, then stop . Otherwise, continue.

3) If ρ_k satisfies both

$$r_1 := \frac{\|\rho_k[\langle g(\tilde{u}^k) - g(\bar{u}^k), T(u^k) - T(\tilde{u}^k)\rangle - \langle g(u^k) - g(\bar{u}^k), T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|}{\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2} \leq \mu^2 \tag{3.2}$$

and

$$r_2 := \frac{\|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|}{\|g(\tilde{u}^k) - g(\bar{u}^k)\|} \leq \nu, \tag{3.3}$$

then go to Step 4; otherwise, continue.

4) Perform an Armijo-like line search via reducing ρ_k

$$\rho_k := \rho_k * \frac{0.8}{\max(r_1, 1)} \tag{3.4}$$

and go to Step 3.

Step 4. Take the new iteration u^{k+1}, by setting

$$g(u^{k+1}) = g(u^k) - \alpha_k d(\tilde{u}^k, \bar{u}^k), \tag{3.5}$$

where

$$\alpha_k = \frac{\langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle}{\|d(\tilde{u}^k, \bar{u}^k)\|^2} \tag{3.6}$$

and

$$d(\tilde{u}^k, \bar{u}^k) := (g(\tilde{u}^k) - g(\bar{u}^k)) - \rho_k(T(\tilde{u}^k) - T(\bar{u}^k)). \tag{3.7}$$

Step 5. Adaptive rule of choosing a suitable ρ_{k+1} as the start prediction step size for the next iteration

1) Prepare a proper ρ_{k+1},

$$\rho_{k+1} := \begin{cases} \rho_k * \tau/r_2 & \text{if } r_2 \leq \eta_1, \\ \rho_k * \tau/r_2 & \text{if } r_2 \geq \eta_2, \\ \rho_k & \text{otherwise.} \end{cases} \tag{3.8}$$

2) Return to Step 2, with k replaced by $k+1$.

If $\varphi(v,u) = \varphi(v), \forall u \in H$, and φ is an indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$ [24], the projection of H onto K and consequently Algorithm 3.1 collapses to the following Algorithm for solving the general variational inequalities, which is due to Bnouhachem and Noor [10].

Algorithm 3.2

Step 1. Given $u^0 \in H$, $\epsilon > 0$, $\rho_0 = 1$, $\nu > 1$, $\mu \in (0, \sqrt{2})$, $\tau \in (0,1)$, $\eta_1 \in (0,\tau)$, $\eta_2 \in (\tau, \nu)$ and let $k = 0$.

Step 2. If $\|r(u^k, 1)\| \leq \epsilon$, then stop. Otherwise, go to Step 3.

Step 3. 1) For a given $u^k \in H$, calculate the two predictors

$$g(\tilde{u}^k) = P_K[g(u^k) - \rho_k T(u^k)],$$

$$g(\bar{u}^k) = P_K[g(\tilde{u}^k) - \rho_k T(\tilde{u}^k)].$$

2) If $\|r(\tilde{u}^k, 1)\| \leq \epsilon$, then stop . Otherwise, continue.

3) If ρ_k satisfies both

$$r_1 := \frac{\|\rho_k[\langle g(\tilde{u}^k) - g(\bar{u}^k), T(u^k) - T(\tilde{u}^k)\rangle - \langle g(u^k) - g(\bar{u}^k), T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|}{\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2} \leq \mu^2$$

and

$$r_2 := \frac{\|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|}{\|g(\tilde{u}^k) - g(\bar{u}^k)\|} \leq \nu,$$

then go to Step 4; otherwise, continue.

4) Perform an Armijo-like line search via reducing ρ_k

$$\rho_k := \rho_k * \frac{0.8}{\max(r_1, 1)}$$

and go to Step 3.

Step 4. Take the new iteration u^{k+1}, by setting

$$g(u^{k+1}) = g(u^k) - \alpha_k d(\tilde{u}^k, \bar{u}^k),$$

where

$$\alpha_k = \frac{\langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle}{\|d(\tilde{u}^k, \bar{u}^k)\|^2}$$

and

$$d(\tilde{u}^k, \bar{u}^k) := (g(\tilde{u}^k) - g(\bar{u}^k)) - \rho_k(T(\tilde{u}^k) - T(\bar{u}^k)).$$

Step 5. Adaptive rule of choosing a suitable ρ_{k+1} as the start prediction step size for the next iteration

1) Prepare a proper ρ_{k+1},

$$\rho_{k+1} := \begin{cases} \rho_k * \tau / r_2 & \text{if } r_2 \leq \eta_1, \\ \rho_k * \tau / r_2 & \text{if } r_2 \geq \eta_2, \\ \rho_k & \text{otherwise.} \end{cases}$$

2) Return to Step 2, with k replaced by $k+1$.

Lemma 3.1 Let u^* be a solution of problem (2.1). For given $u^k \in H$, let $\tilde{u}^k, \bar{u}^k$ be the predictors produced by (3.1a) and (3.1b), then we have

$$\langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle \geq (2 - \mu^2)\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2. \tag{3.9}$$

Proof. Note that $g(\tilde{u}^k) = J_{\varphi(u^k)}[g(u^k) - \rho_k T(u^k)]$, $g(\bar{u}^k) = J_{\varphi(u^k)}[g(\tilde{u}^k) - \rho_k T(\tilde{u}^k)]$, we can apply (2.14) with $v = g(u^k) - \rho_k T(u^k)$, $w = g(\tilde{u}^k) - \rho_k T(\tilde{u}^k)$ to obtain

$$\langle g(u^k) - \rho_k T(u^k) - (g(\tilde{u}^k) - \rho_k T(\tilde{u}^k)), g(\tilde{u}^k) - g(\bar{u}^k)\rangle \geq \|g(\tilde{u}^k) - g(\bar{u}^k)\|^2.$$

By some manipulations, we have

$$\langle g(u^k) - g(\tilde{u}^k), g(\tilde{u}^k) - g(\bar{u}^k)\rangle \geq \|g(\tilde{u}^k) - g(\bar{u}^k)\|^2 + \rho_k\langle g(\tilde{u}^k) - g(\bar{u}^k), T(u^k) - T(\tilde{u}^k)\rangle.$$

Then, we obtain

$$\begin{aligned} \langle g(u^k) - g(\tilde{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle &= \langle g(u^k) - g(\tilde{u}^k), g(\tilde{u}^k) - g(\bar{u}^k)\rangle - \rho_k\langle g(u^k) - g(\tilde{u}^k), T(\tilde{u}^k) - T(\bar{u}^k)\rangle \\ &\geq \|g(\tilde{u}^k) - g(\bar{u}^k)\|^2 + \rho_k\langle g(\tilde{u}^k) - g(\bar{u}^k), T(u^k) - T(\tilde{u}^k)\rangle \\ &\quad -\rho_k\langle g(u^k) - g(\tilde{u}^k), T(\tilde{u}^k) - T(\bar{u}^k)\rangle. \end{aligned} \tag{3.10}$$

Using (3.10), (3.2) and the definition of $d(\tilde{u}^k, \bar{u}^k)$, we get

$$\begin{aligned} \langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle &= \langle g(u^k) - g(\tilde{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle + \langle g(\tilde{u}^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle \\ &\geq \|g(\tilde{u}^k) - g(\bar{u}^k)\|^2 + \rho_k\langle g(\tilde{u}^k) - g(\bar{u}^k), T(u^k) - T(\tilde{u}^k)\rangle \\ &\quad -\rho_k\langle g(u^k) - g(\tilde{u}^k), T(\tilde{u}^k) - T(\bar{u}^k)\rangle + \|g(\tilde{u}^k) - g(\bar{u}^k)\|^2 \\ &\quad -\rho_k\langle g(\tilde{u}^k) - g(\bar{u}^k), T(\tilde{u}^k) - T(\bar{u}^k)\rangle \\ &\geq (2 - \mu^2)\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2. \end{aligned}$$

Hence, (3.9) holds and the proof is completed. $\square$

Now, we mainly focus on investigating the convergence of Algorithm 3.1. The following theorem plays a crucial role in the convergence of Algorithm 3.1.

Theorem 3.1 Let u^* be a solution of problem (2.1) and let $g(u^{k+1})$ be the sequence obtained from algorithm 3.1. Then u^k is bounded and

$$\|g(u^{k+1}) - g(u^*)\|^2 \leq \|g(u^k) - g(u^*)\|^2 - \frac{(2-\mu^2)^2}{(1+\nu)^2}\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2. \tag{3.11}$$

Proof. For any $u^* \in S^*$ solution of problem (2.1), we have

$$\langle \rho_k T(u^*), g(\bar{u}^k) - g(u^*)\rangle + \rho_k\varphi(g(\bar{u}^k), g(u^*)) - \rho_k\varphi(g(u^*), g(u^*)) \geq 0.$$

Using the g-pseudomonotonicity of T, we obtain

$$\langle \rho_k T(\bar{u}^k), g(\bar{u}^k) - g(u^*)\rangle + \rho_k\varphi(g(\bar{u}^k), g(u^*)) - \rho_k\varphi(g(u^*), g(u^*)) \geq 0. \tag{3.12}$$

Substituting $w = g(\tilde{u}^k) - \rho_k T(\tilde{u}^k)$ and $v = g(u^*)$ into (2.10), we get

$$\langle g(\tilde{u}^k) - \rho_k T(\tilde{u}^k) - g(\bar{u}^k), g(\bar{u}^k) - g(u^*)\rangle + \rho_k \varphi(g(u^*), g(\bar{u}^k)) - \rho_k \varphi(g(\bar{u}^k), g(\bar{u}^k)) \geq 0. \quad (3.13)$$

Adding (3.12) and (3.13), and using the definition of $d(\tilde{u}^k, \bar{u}^k)$, we have

$$\langle d(\tilde{u}^k, \bar{u}^k), g(\bar{u}^k) - g(u^*)\rangle \geq 0. \quad (3.14)$$

Since $u^* \in H$ be a solution of problem (2.1), then

$$\begin{aligned} \|g(u^{k+1}) - g(u^*)\|^2 &= \|g(u^k) - g(u^*) - \alpha_k d(\tilde{u}^k, \bar{u}^k)\|^2 \\ &= \|g(u^k) - g(u^*)\|^2 - 2\alpha_k \langle g(u^k) - g(u^*), d(\tilde{u}^k, \bar{u}^k)\rangle \\ &\quad + \alpha_k^2 \|d(\tilde{u}^k, \bar{u}^k)\|^2. \end{aligned} \quad (3.15)$$

Adding (3.14) (multiplied by $2\alpha_k$) to (3.15) and using (3.6), we get

$$\begin{aligned} \|g(u^{k+1}) - g(u^*)\|^2 &\leq \|g(u^k) - g(u^*)\|^2 - 2\alpha_k \langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle + \alpha_k^2 \|d(\tilde{u}^k, \bar{u}^k)\|^2 \\ &= \|g(u^k) - g(u^*)\|^2 - \alpha_k \langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle \\ &\leq \|g(u^k) - g(u^*)\|^2 - \alpha_k (2 - \mu^2) \|g(\tilde{u}^k) - g(\bar{u}^k)\|^2 \end{aligned} \quad (3.16)$$

where the last inequality follows from (3.9).

Recalling the definition of $d(\tilde{u}^k, \bar{u}^k)$ (see (3.7)) and applying Criterion (3.3), it is easy to see that

$$\|d(\tilde{u}^k, \bar{u}^k)\|^2 \leq (\|g(\tilde{u}^k) - g(\bar{u}^k)\| + \|\rho_k (T(\tilde{u}^k) - T(\bar{u}^k))\|)^2 \leq (1+\nu)^2 \|g(\tilde{u}^k) - g(\bar{u}^k)\|^2. \quad (3.17)$$

Moreover, by using (3.9) together with (3.17), we get

$$\alpha_k = \frac{\langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle}{\|d(\tilde{u}^k, \bar{u}^k)\|^2} \geq \frac{2 - \mu^2}{(1+\nu)^2} > 0, \qquad \mu \in (0, \sqrt{2}). \quad (3.18)$$

Substituting (3.18) in (3.16), we get the assertion of this theorem. Since $\gamma \in [1, 2)$ and $\mu \in (0, \sqrt{2})$ we have

$$\|g(u^{k+1}) - g(u^*)\| \leq \|g(u^k) - g(u^*)\| \leq \ldots \leq \|g(u^0) - g(u^*)\|.$$

Since g is homeomorphism and from the above inequality, it is easy to verify that the sequence $\{u^k\}$ is bounded. $\square$

We now present the convergence result of Algorithm 3.1.

Theorem 3.2 If $\inf_{k=0}^{\infty} \rho_k := \rho > 0$, then any cluster point of the sequence $\{\tilde{u}^k\}$ generated by Algorithm 3.1 is a solution of problem (2.1).

Proof. It follows from (3.11) that

$$\lim_{k\to\infty} \|g(\tilde{u}^k) - g(\bar{u}^k)\| = 0.$$

Since the sequence $\{u^k\}$ is bounded, $\{\tilde{u}^k\}$ is also bounded, it has at least a cluster point. Let u^∞ be a cluster point of $\{\tilde{u}^k\}$ and the subsequence $\{\tilde{u}^{k_j}\}$ converges to u^∞. Using the continuity of $r(u, \rho)$ and inequality (2.12), it follows that

$$\|r(u^\infty, \rho)\| = \lim_{k_j\to\infty} \|r(\tilde{u}^{k_j}, \rho)\| \le \lim_{k_j\to\infty} \|r(\tilde{u}^{k_j}, \rho_{k_j})\| = \lim_{k_j\to\infty} \|g(\tilde{u}^{k_j}) - g(\bar{u}^{k_j})\| = 0.$$

This means that u^∞ is a solution of problem (2.1). $\square$

Let $g(w^k) = g(u^k) - \alpha_k \rho_k T(\tilde{u}^k)$. For a positive constant τ, we consider

$$g(u^{k+1}) = g(u^k) - \tau(g(u^k) - g(w^k)).$$

Here the positive constant τ can be viewed as a step along the direction $-(g(u^k) - g(w^k))$. We use the fixed-point formulation to suggest the following iterative method.

Algorithm 3.3

Step 1. Given $u^0 \in H$, $\epsilon > 0$, $\rho_0 = 1$, $\nu > 1$, $\mu \in (0, \sqrt{2})$, $\tau \in (0, 1)$, $\eta_1 \in (0, \tau)$, $\eta_2 \in (\tau, \nu)$ and let $k = 0$.

Step 2. If $\|r(u^k, 1)\| \le \epsilon$, then stop. Otherwise, go to Step 3.

Step 3. 1) For a given $u^k \in H$, calculate the two predictors

$$g(\tilde{u}^k) = J_{\varphi(u^k)}[g(u^k) - \rho_k T(u^k)],$$

$$g(\bar{u}^k) = J_{\varphi(u^k)}[g(\tilde{u}^k) - \rho_k T(\tilde{u}^k)].$$

2) If $\|r(\tilde{u}^k, 1)\| \le \epsilon$, then stop . Otherwise, continue.

3) If ρ_k satisfies both

$$r_1 := \frac{\|\rho_k[\langle g(\tilde{u}^k) - g(\bar{u}^k), T(u^k) - T(\tilde{u}^k)\rangle - \langle g(u^k) - g(\bar{u}^k), T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|}{\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2} \leq \mu^2$$

and

$$r_2 := \frac{\|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|}{\|g(\tilde{u}^k) - g(\bar{u}^k)\|} \leq \nu,$$

then go to Step 4; otherwise, continue.

4) Perform an Armijo-like line search via reducing ρ_k

$$\rho_k := \rho_k * \frac{0.8}{\max(r_1, 1)}$$

and go to Step 3.

Step 4. Compute

$$g(w^k) = g(u^k) - \alpha_k d(\tilde{u}^k, \bar{u}^k)),$$

where

$$\alpha_k = \frac{\langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle}{\|d(\tilde{u}^k, \bar{u}^k)\|^2}$$

and

$$d(\tilde{u}^k, \bar{u}^k) := (g(\tilde{u}^k) - g(\bar{u}^k)) - \rho_k(T(\tilde{u}^k) - T(\bar{u}^k)).$$

Step 5. For $\tau > 0$, the new iterate $u^{k+1}(\tau)$ is defined by

$$g(u^{k+1}(\tau)) = g(u^k) - \tau(g(u^k) - g(w^k)). \tag{3.19}$$

Step 6. Adaptive rule of choosing a suitable ρ_{k+1} as the start prediction step size for the next iteration

1) Prepare a proper ρ_{k+1},

$$\rho_{k+1} := \begin{cases} \rho_k * \tau/r_2 & \text{if } r_2 \leq \eta_1, \\ \rho_k * \tau/r_2 & \text{if } r_2 \geq \eta_2, \\ \rho_k & \text{otherwise.} \end{cases}$$

2) Return to Step 2, with k replaced by $k+1$.

If $\varphi(v,u) = \varphi(v), \forall u \in H$, and φ is an indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$ [24], the projection of H onto K and Consequently Algorithm 3.3 reduces to Algorithm 3.4 for solving the general variational inequalities (2.4).

Algorithm 3.4

Step 1. Given $u^0 \in H$, $\epsilon > 0$, $\rho_0 = 1$, $\nu > 1$, $\mu \in (0, \sqrt{2})$, $\tau \in (0,1)$, $\eta_1 \in (0,\tau)$, $\eta_2 \in (\tau, \nu)$ and let $k = 0$.

Step 2. If $\|r(u^k, 1)\| \leq \epsilon$, then stop. Otherwise, go to Step 3.

Step 3. 1) For a given $u^k \in H$, calculate the two predictors

$$g(\tilde{u}^k) = P_K[g(u^k) - \rho_k T(u^k)],$$

$$g(\bar{u}^k) = P_K[g(\tilde{u}^k) - \rho_k T(\tilde{u}^k)].$$

2) If $\|r(\tilde{u}^k, 1)\| \leq \epsilon$, then stop . Otherwise, continue.

3) If ρ_k satisfies both

$$r_1 := \frac{\|\rho_k[\langle g(\tilde{u}^k) - g(\bar{u}^k), T(u^k) - T(\tilde{u}^k)\rangle - \langle g(u^k) - g(\bar{u}^k), T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|}{\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2} \leq \mu^2$$

and

$$r_2 := \frac{\|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|}{\|g(\tilde{u}^k) - g(\bar{u}^k)\|} \leq \nu,$$

then go to Step 4; otherwise, continue.

4) Perform an Armijo-like line search via reducing ρ_k

$$\rho_k := \rho_k * \frac{0.8}{\max(r_1, 1)}$$

and go to Step 3.

Step 4. Compute

$$g(w^k) = g(u^k) - \alpha_k d(\tilde{u}^k, \bar{u}^k)),$$

where

$$\alpha_k = \frac{\langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle}{\|d(\tilde{u}^k, \bar{u}^k)\|^2}$$

and

$$d(\tilde{u}^k, \bar{u}^k) := (g(\tilde{u}^k) - g(\bar{u}^k)) - \rho_k(T(\tilde{u}^k) - T(\bar{u}^k)).$$

Step 5. For $\tau > 0$, the new iterate $u^{k+1}(\tau)$ is defined by

$$g(u^{k+1}(\tau)) = g(u^k) - \tau(g(u^k) - g(w^k)).$$

Step 6. Adaptive rule of choosing a suitable ρ_{k+1} as the start prediction step size for the next iteration

1) Prepare a proper ρ_{k+1},

$$\rho_{k+1} := \begin{cases} \rho_k * \tau/r_2 & \text{if } r_2 \le \eta_1, \\ \rho_k * \tau/r_2 & \text{if } r_2 \ge \eta_2, \\ \rho_k & \text{otherwise.} \end{cases}$$

2) Return to Step 2, with k replaced by $k+1$.

How to choose a suitable step length $\tau > 0$ to force convergence will be discussed later. We now consider the criteria of τ, which ensures that $g(u^{k+1}(\tau))$ is closer to $g(u^*)$ than $g(u^k)$. For this purpose, we define

$$\Gamma(\tau) := \|g(u^k) - g(u^*)\|^2 - \|g(u^{k+1}(\tau)) - g(u^*)\|^2. \tag{3.20}$$

Lemma 3.2 Let $u^* \in S^*$ and $g(w^k) = g(u^k) - \alpha_k d(\tilde{u}^k, \bar{u}^k))$. Then we have

$$\Gamma(\tau) = \tau\{\|g(u^k) - g(w^k)\|^2 + \Upsilon(\alpha_k)\} - \tau^2\|g(u^k) - g(w^k)\|^2, \tag{3.21}$$

where

$$\Upsilon(\alpha_k) := \|g(u^k) - g(u^*)\|^2 - \|g(w^k) - g(u^*)\|^2. \tag{3.22}$$

Proof. It follows from (3.19) that

$$\begin{aligned} \Gamma(\tau) &= \|g(u^k) - g(u^*)\|^2 - \|g(u^k) - \tau(g(u^k) - g(w^k)) - g(u^*)\|^2 \\ &= 2\tau\langle g(u^k) - g(u^*), g(u^k) - g(w^k)\rangle - \tau^2\|g(u^k) - g(w^k)\|^2 \\ &= 2\tau\{\|g(u^k) - g(w^k)\|^2 - \langle g(u^*) - g(w^k), g(u^k) - g(w^k)\rangle\} \\ &\quad -\tau^2\|g(u^k) - g(w^k)\|^2 \end{aligned} \tag{3.23}$$

Using the following identity

$$\langle g(u^*) - g(w^k), g(u^k) - g(w^k)\rangle = \frac{1}{2}\left(\|g(w^k) - g(u^*)\|^2 - \|g(u^k) - g(u^*)\|^2\right) + \frac{1}{2}\|g(u^k) - g(w^k)\|^2,$$

and the notation of $\Upsilon(\alpha_k)$, we obtain (3.21), the required result. $\Box$

Using (3.16)(by setting $g(w^k) = g(u^{k+1})$) and (3.21), we get

$$\Gamma(\tau) \geq \Lambda(\tau), \tag{3.24}$$

where

$$\Lambda(\tau) = \tau\{\|g(u^k) - g(w^k)\|^2 + \alpha_k\langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle\} - \tau^2\|g(u^k) - g(w^k)\|^2.$$

The above inequality tells us how to choose a suitable τ_k. Since $\Lambda(\tau_k)$ is a quadratic function of τ_k and it reaches its maximum at

$$\tau_k^* = \frac{\|g(u^k) - g(w^k)\|^2 + \alpha_k\langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle}{2\|g(u^k) - g(w^k)\|^2}$$

and

$$\Lambda(\tau_k^*) = \frac{\tau_k^*\{\|g(u^k) - g(w^k)\|^2 + \alpha_k\langle g(u^k) - g(\bar{u}^k), d(\tilde{u}^k, \bar{u}^k)\rangle\}}{2}.$$

Then, from Lemma 3.1 and (3.18), we get

$$\begin{aligned} \tau_k^* &\geq \frac{\|g(u^k) - g(w^k)\|^2 + \frac{(2-\mu^2)^2}{(1+\nu)^2}\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2}{2\|g(u^k) - g(w^k)\|^2} \\ &\geq \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} \Lambda(\tau_k^*) &\geq \frac{\tau_k^*(2-\mu^2)^2}{2(1+\nu)^2}\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2 \\ &\geq \frac{(2-\mu^2)^2}{4(1+\nu)^2}\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2. \end{aligned} \tag{3.25}$$

Then, from (3.20), (3.24) and (3.25), we have

$$\begin{aligned} \|g(u^{k+1}) - g(u^*)\|^2 &= \|g(u^k) - g(u^*)\|^2 - \Gamma(\tau_k^*) \\ &\leq \|g(u^k) - g(u^*)\|^2 - \frac{(2-\mu^2)^2}{4(1+\nu)^2}\|g(\tilde{u}^k) - g(\bar{u}^k)\|^2. \end{aligned}$$

The convergence of Algorithm 3.2 can be proved by similar arguments as Algorithm 3.1. Hence the proof is omitted.

Remark 3.1 If $\tau_k^* = 1$ Algorithm 3.4 reduces to Algorithm 3.2. Since τ_k^* is to maximize the profit function $\Lambda(\tau)$, we have

$$\Lambda(\tau_k^*) \geq \Lambda(1). \tag{3.26}$$

Inequalities (3.24) and (3.26) show theoretically that Algorithm 3.4 is expected to make more progress than Algorithm 3.2 at each iteration, and so it explains theoretically that Algorithm 3.4 outperforms Algorithm 3.2.

4 Conclusions

In this paper, we suggest and analyze two new methods for solving general mixed quasi variational inequalities, which can be viewed as a refinement and improvement of some existing resolvent methods and projection descent methods. It is easy to verify that Algorithm 3.1 and Algorithm 3.3 include some existing methods (e.g. [4, 5, 6]) as special cases. Therefore, the new algorithms are expected to be widely applicable.

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