Regularity of a minimizer for optimal shape problem for the first eigenvalue of the p-Laplacian operator

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Abstract

In this paper, we study the regularity of a minimizer for the optimal shape problem for the first eigenvalue of the p-Laplace operator. Using the associated variational problem, we study the regularity of the optimal first eigenfunction corresponding to the optimal shape problem for p-Laplacian operator with volume and inclusion constraints. And we prove the equivalence between the associated variational and penalized problems provided the penalization parameter λ is large enough. We study also the minimizers of such family of penalized problem showing they are Hölder continuous.

Mathematics Subject Classification(2010),35R37,47J10,35Q80,49Q10,49K20
Keywords: p-Laplacian, Shape optimization, Nonlinear eigenvalue

1 Introduction

Let D be a bounded open set of $\mathbb{R}^N$. For all open subset Ω of D, we denote by $\lambda_1(\Omega)$ the first eigenvalue of the p-Laplacian operator in Ω, with homogeneous boundary conditions, and by u_Ω a normalized eigenfunction, that is

$$\begin{cases} -\Delta_p u_\Omega &= \lambda_1(\Omega)|u_\Omega|^{p-2}u_\Omega \quad in \quad \Omega \\ u_\Omega &= 0 \quad on \quad \partial\Omega \\ \int_\Omega |u_\Omega|^p &= 1 \end{cases} \tag{1}$$

The goal of this paper is to study the regularity of the optimal shapes of the the following shape optimization problem where $c \in]0, |D|[$ and $|D|$ denotes the Lebesgue measure of $|D|$.

$$\begin{cases} \Omega^* \text{ open set }, \ \Omega^* \subset D, \ |\Omega^*| = c \\ \lambda_1(\Omega^*) = \min\{\lambda_1(\Omega), \quad \Omega \in \mathcal{O}_c\} \end{cases} \tag{2}$$

where $\mathcal{O}_c$ is the class of admissible sets defined by

$$\mathcal{O}_c = \{\Omega, \text{ open set}, \ \Omega \subset D \text{ and } |\Omega| = c\}$$

***AMO-Advanced Modeling and Optimization.ISSN:1841-4311**

and

$$\left\{\begin{array}{lll} \lambda_1(\Omega) & = & J(u_\Omega) = \int_\Omega |\nabla u_\Omega|^p \\ & = & \min\{\int_\Omega |\nabla v|^p,\ v \ \in\ W_0^{1,p}(\Omega), \text{and} \int_\Omega |v|^p = 1\} \end{array}\right. \tag{3}$$

so that u_Ω is the unique solution to the problem (1) which reaches the minimum (3), see [17],[27].

One can prove using the Schwarz symmetrization method that any ball $B \subset D$ with the prescribed measure $|B| = c$ is solution to the problem (2), see [29] . The shape optimization problem (2) does not always have a solution in $\mathcal{O}_c$. The existence result when $\mathcal{O}_c$ consists in all quasi open subsets of D is well known from the general result of Buttazzo and Dalmaso see [11]. For $p > 2$, we introduce the class $\mathcal{A}_p(D)$ of p-quasi open sets in D and the γ_p convergence and show there exists a minimizer among the quasi open sets, see [29].But generally, we cannot say more about regularity of the optimal quasi open set.

The idea is to introduce a penalty term depending on c and to consider a variational problem. The advantage is to involve only the state function and not the optimal shape, see Alt and Caffarelli [2] and Alt , Caffarelli and Friedman [3].

This approach was used by C. Bandle and A. Wagner [4] for a variational problem under a constraint on the mass which is motivated by the torsional rigidity and torsional creep.They treat instead a problem without constraint but with a penalty term. And they established the existence of a Lipschitz continuous minimizer and prove qualitative properties of the optimal shape.

In [8], T.Briancon et al proved: if $u \ \in\ H_0^1(D)$ is a solution of

$$\lambda_m = \int_\Omega |\nabla u|^2 dx = \min\{\int_\Omega |\nabla v|^2,\ v \ \in\ H_0^1(D), \int_D |v|^2 = 1 \text{ and } |\Omega_v| \ \leq\ m, m \ \in\ (0, |D|)\}$$

then u is Lipschitz continuous on D. And the shape optimization problem

$$\lambda_1(\Omega^*) = \min\{\lambda_1(\Omega), \Omega \text{ quasi open set}, \Omega \subset\ D \text{ and } |\Omega| \leq\ m\}$$

has a solution Ω^* with $|\Omega^*| = m$ and which is at least an open subset of D whose corresponding eigenfunction is locally Lipschitz continuous.

In [9], Briancon and Lamboley considering the shape optimization problem $\lambda_1(\Omega^*) = \min\{\lambda_1(\Omega),\ \Omega$ open set, $\Omega \subset\ D$ and $|\Omega| = m\}$ proved regularity properties of the boundary of the optimal shape Ω^* in any case and any dimension.

In this paper, we are going to use a variational approach to solve the problem (2). And we replace the shape optimization problem by the following variational problem

$$\left\{\begin{array}{lll} \text{Find } u \ \in\ \mathbb{V}, & \text{such that} & \forall\ v \ \in\ \mathbb{V} \\ & J(u) \ \leq\ J(v) & \end{array}\right. \tag{4}$$

where the class of admissible functions is given by

$$\mathbb{V} := \{v \in W_0^{1,p}(D), \quad \int_D |v|^p = 1 \text{ and } |\Omega_v| = c\}$$

with $\Omega_v = \{x \in D, \ v(x) > 0\}$.

The objective of this paper is to prove that problem (4) admits at least one continuous minimizer u and consequently that Ω_u solves the problem (2). Moreover when $p > N$, the Hölder continuity of any minimizer is a consequence of the Sobolev embedding theorem $W_0^{1,p}(D) \hookrightarrow \mathcal{C}^{0,\alpha}(D)$ with $\alpha = 1 - \frac{N}{p}$. The main difficulty is the regularity of a minimizer fails using Sobolev embedding when $1 < p \leq N$.

The structure of this paper is organized as follows: In section 2 we give some auxiliary results. In section 3 we give the main result and its application to shape optimization problem. In section 4, we show the constrained problem is equivalent to a penalized problem and we prove that such functions minimize the initial problem (4), provided the penalization parameter λ is large enough. In the last section, we study the Hölder continuity of the minimizers of a family of penalized functionals $J_\lambda, \lambda > 0$.

2 Auxiliary results

Since we are interested in the case $1 < p \leq N$, we shall use some known properties for functions in $W_0^{1,p}(\Omega)$. From Sobolev embedding theorem and the Gagliardo-Nirenberg inequality see for example [18, 27, 28], an application of Hölder inequality yields

$$||u||_{L^s(\Omega)} \leq C|\Omega|^{\frac{1}{s} - \frac{N-p}{Np}} ||\nabla u||_{L^p(\Omega)},$$

which is true for any $u \in W_0^{1,p}(\Omega)$ and for any open set $\Omega \subset D$ and where $1 < s < \frac{Np}{N-p}$, if $p < N$ and the constant C depending only on N, s and p. If $s = p$, we get the following Poincaré-type inequality:

$$||u||_{L^p(\Omega)} \leq C_0|\Omega|^{\frac{1}{N}} ||\nabla u||_{L^p(\Omega)},$$

where $C_0 = C_0(N)$.

3 Main results

The difficulty we have to overcome in order to prove the existence result of the problem (4) is the fact that $\mathbb{V}$ is not closed with respect to the weak topology of $W_0^{1,p}(D)$.

We denote $\mathbb{V}_0$ the class of admissible functions is given by

$$\mathbb{V}_0 := \{v \in W_0^{1,p}(D), \quad \int_D |v|^p = 1 \text{ and } |\Omega_v| \leq c\}$$

with $\Omega_v = \{x \in D,\ v(x) > 0\}$.
And

$$\lambda_c = \lambda_1(\Omega^*) = J(u_{\Omega^*}) = J(u) = \int_\Omega |\nabla u|^p dx = \min\{\int_\Omega |\nabla v|^p,\ v \in \mathbb{V}_0\}$$

However, we consider the following relaxed version

$$\begin{cases} \text{Find } u \in \mathbb{V}_0, \quad \text{such that} \quad \forall\ v \in \mathbb{V}_0 \\ \qquad\qquad J(u) \leq J(v) \end{cases} \tag{5}$$

The set $\mathbb{V}$ is included in $\mathbb{V}_0$, and it is shown in [12]. Moreover we have the following lemmas:

Lemma 3.1 *The class $\mathbb{V}_0$ is weakly closed in $W_0^{1,p}(D)$.*

That is if a sequence of functions $v_n \in \mathbb{V}_0$ converges in weak topology of $W_0^{1,p}(D)$ to $v \in W_0^{1,p}(D)$, then we obtain $v \in \mathbb{V}_0$. We have also

Lemma 3.2 *The set $\mathbb{V}$ is dense in $\mathbb{V}_0$; that is:*

$\forall\ v \in \mathbb{V}_0, \forall\ n \in \mathbb{N},$ *there exists $v_n \in \mathbb{V}$ such that : v_n converges to v in $W_0^{1,p}(D)$.*

Proof of Lemma 3.2 see [21].
We have the following result:

Proposition 3.1 *The problem (5) admits at least one solution.*

Proof of Proposition (3.1)
Thanks to inequality (5) we obtain for any $v \in \mathbb{V}_0$,
$J(v) = \int_\Omega |\nabla v|^p dx, > \frac{1}{C_0|\Omega|^{\frac{1}{N}}}$, then $J(v) > 0$ this implies that $inf\{J(v), v \in \mathbb{V}_0\} > -\infty$. There exists a minimizing sequence (u_n) which converges to $\alpha = inf\{J(v), v \in \mathbb{V}_0\}$. We see that the sequence (u_n) is bounded in $W_0^{1,p}(D)$. There exists $u \in W_0^{1,p}(D)$ and a subsequence still denoted (u_n) such that u_n converges weakly on u in $W_0^{1,p}(D)$. Using the Rellich -Kondrachov theorem , (u_n) converges on u in $L^p(D)$ and a.e in D. We have $J(u) \leq \lim_{n\longrightarrow+\infty} inf J(u_n)$ and also $|\Omega_u| \leq \lim_{n\longrightarrow+\infty} inf|\Omega_n| \leq c$ ∎

Moreover, the following result also holds

Lemma 3.3 *If u is a solution of (5) such that $|\Omega_u| < c$, then $u = u_D$.*

Proof of lemma (3.3)
Let $x_0 \in D$, there exists a small ball $B(x_0, r_0) \subset D$ of radius r_0 verifying $|B(x_0, r_0)| < c - |\Omega_u|$ such that $r_0 < (\frac{c-|\Omega_u|}{W_N})^{1/N}$, where $W_N = |B(O,1)|$. For all $t \in \mathbb{R}$ and for all $\phi \in \mathcal{C}_0^\infty(B(x_0, r))$, we have $\frac{(u+t\phi)}{\int_D |(u+t\phi)|^p dx}$ belongs to $\mathbb{V}_0$. We obtain

$$\int_D |\nabla u|^p dx \leq \frac{\int_D |\nabla(u+t\phi)|^p dx}{\int_D |(u+t\phi)|^p dx}.$$

By differentiating with respect to t at $t = 0$ implies
$\int_D |\nabla u|^{p-2} \nabla u \nabla \phi dx = \lambda_c \int_D |u|^{p-2} u\phi = \lambda_c \int_D |u|^{p-1}\phi$, in this case, $u = u_D$ since x_0 is an arbitrary point in D. ∎

From a mathematical point of view, the problem (5) is easier to study. Fortunately we have the following relation.

Lemma 3.4 *We assume the following condition holds*

$$\begin{cases} \text{there does not exist} \quad u \in W_0^{1,p}(D) \quad \text{with } |\Omega_u| < c \text{ such that} \\ \qquad\qquad -\Delta_p u = \lambda_c |u|^{p-1} \text{ in } D \end{cases} \tag{6}$$

Then the problems (4) and (5) are equivalent.

Proof of lemma (3.4)
From the density of $\mathbb{V}$ in $\mathbb{V}_0$, u solution of (4) implies u solution of the problem (5). Conversely, let be u a solution of the problem (5) and $|\Omega_u| = c'$. If u were not a solution of the problem (4), we should have $c' < c$. And for all $\phi \in W_0^{1,p}(D)$, with $|\Omega_\phi| \leq c - c'$, for all $t \in \mathbb{R}$, $\frac{(u+t\phi)}{\int_D |(u+t\phi)|^p dx}$ belongs to $\mathbb{V}_0$. Therefore we get

$$\int_D |\nabla u|^p dx \leq \frac{\int_D |\nabla(u+t\phi)|^p dx}{\int_D |(u+t\phi)|^p dx}.$$

By differentiating with respect to t at $t = 0$ implies $\int_D |\nabla u|^{p-2} \nabla u \nabla \phi dx = \lambda_c \int_D |u|^{p-2} u\phi = \lambda_c \int_D |u|^{p-1}\phi$, so that we can conclude that the condition (6) does not hold. ■

The following result indicates the relationship between problem (4) and the problem (5).

Proposition 3.2 *Assume that (6) holds. Then the following holds true:*

(a) Any solution of (5) is also a solution of (4).

(b) The problem (4) admits at least one solution.

(c) Any solution to (4) is also a solution to (5)

Proof of proposition (3.2)
(a) Let u be a solution of (5). By Lemma (3.4) we see that $|\Omega_u| = c$ *i.e* $u \in \mathbb{V}$. But since $\mathbb{V} \subset \mathbb{V}_0$, we have also $J(u) \leq J(v)$ for any $v \in \mathbb{V}$. Thus u is a solution of (4).
(b) This is an easy consequence of (a) and the proposition (3.1).
(c) Let u be a solution of (4) and by the proposition (3.1), the problem (5) has at least one solution z. From (a) we have that z is also a solution of (4), and therefore $J(u) = J(z) \leq J(v)$ for any $v \in \mathbb{V}_0$. That is u is indeed a solution to (5) because $u \in \mathbb{V} \subset \mathbb{V}_0$. ■

For the rest of the paper we shall assume the following condition.

$$\begin{cases} \text{Assume (6) holds} \quad \text{and that} \quad |u|^{p-1} \in L^q(D) \text{ where } q \geq \frac{p}{p-1} \\ \qquad\qquad\qquad\qquad\qquad \text{and } q > N \end{cases} \tag{7}$$

Remark 3.1 • *In the case $p = 2$, $N \geq 2$, and D fixed domain subset of $\mathbb{R}^N$, non necessarily bounded, we have $\lambda_c u^{p-1} = \lambda_c u = f$. Assuming f belongs to $L^\infty(D) \cap L^2(D)$, T.Briancon et al [8] proved the state function u is locally Lipschitz continuous.*

- *In the case $p = N = 2$ and $D = \mathbb{R}^2$ we have $\lambda_c u^{p-1} = \lambda_c u = f$. In [12], M.Crouzeix proved the Lipschitz regularity of any solution to (4) for f belonging to $L^\infty(\mathbb{R}^2)$ with compact support K satisfying $|K| < c$. In this case the problem arises in electromagnetic shaping of molten metals without surface tension.*

- *In the case $p = 2$, $N \geq 2$, and D sufficiently smooth domain (non necessarily bounded) and $\lambda_c u^{p-1} = \lambda_c u = f$. In [21], M.Hayouni proved the Lipschitz continuity of a solution to (4) which doesn't change its sign for f belonging to $L^q(D)$ with $q > N$.*

- *The validity of assumption (7) comes from [8] ,[12] and [21]. In our case we assume that $|u|^{p-1} \in L^q(D)$ with $q > N$ to expect the state function u to be locally Hölder-continuous. In the other way, we have $u \in W_0^{1,p}(D)$ then $u \in L^p(D)$ implies that $u^{p-1} \in L^{\frac{p}{p-1}}(D)$. If $|u|^{p-1} \in L^q(D)$, q large enough i.e $q > N$ this implies that $q \geq \frac{p}{p-1}$ if not we have $q < \frac{p}{p-1} \leq p \leq N$. This assumption play an important role in the estimation of $\int |\nabla u|^p dx$, see proposition (5.1).*

Our main result is the following. This result is a direct consequence of theorem (5.1) and proposition (4.2) (see section 4.)

Theorem 3.1 *If condition (7) is satisfied then any solution of (4) (equivalently(5)) is locally Hölder-continuous.*

We have the following shape optimization result if condition (6) holds.

Proposition 3.3 *If the condition (7) is satisfied then the shape optimization problem (2) has at least one solution.*

Proof of proposition(3.3)
According to proposition (3.2) and theorem (3.1), problem (5) admits at least one continuous solution u, which is also a solution to (4). We deduce that Ω_u is an open set satisfying $|\Omega_u| = c$ that is $\Omega_u \in \mathcal{O}_c$. We also know $u \in W_0^{1,p}(\Omega_u)$ and $\lambda_1(\Omega_u) = J(u)$. Indeed, by the optimality of u we have $J(u) \leq J(v)$ for any $v \in W_0^{1,p}(\Omega_u)$. On the other hand for any $\Omega \in \mathcal{O}_c$, we have $u \in \mathbb{V}_0$. Since u solves problem (5) we obtain for all $\Omega \in \mathcal{O}_c$, $\lambda_1(\Omega_u) = J(u) \leq \lambda_1(\Omega)$. Finally Ω_u is a solution of the shape optimization problem (2). ■

Remark 3.2 *If the condition (6) does not hold then the shape optimization problem (2) has an infinite number of solution, see [6, 21].*

Remark 3.3 *The existence of optimal quasi-open set doesn't imply generally the existence of an optimal set. It is possible to construct some examples where the optimal quasi-open set is not an open set see [6],[21]. Therefore the shape optimization problem (2) has no solution. This is why we need some assumption to* $|u|^{p-1}$ *to be more regular than* $|u|^{p-1}$ *belonging only to the dual space of* $W_0^{1,p}(D)$.

Remark 3.4 *All we need to prove that* Ω_u *is a solution of the problem (2) i.e* Ω_u *is an open set. We refer to [7],[8] ,[12] and [21] where the study of the regularity of the boundary is made assuming local Lipschitz continuity of the state function.*

4 Equivalence with penalized problem

The interest of penalization is that the admissible set of functions is the whole space $W_0^{1,p}(D)$, moreover the value of $J(v)$ doesn't change for v belongs to $\mathbb{V}_0$ *i.e* $J(v) = J_\lambda(v)$ for any $v \in \mathbb{V}_0$.

According to [2, 3, 8], [16] and [31], we introduce a new problem, by replacing the volume constraint on the volume of the support of the admissible functions and the normalization condition by two penalization terms.

We have the following penalized variational problem

$$\begin{cases} \text{Find } u \in W_0^{1,p}(D), & \text{such that} \quad \forall\ v \in W_0^{1,p}(D) \\ & J_\lambda(u) \leq J_\lambda(v), \end{cases} \tag{8}$$

where for $\lambda > 0$, the functional J_λ is defined as:

$$J_\lambda(v) := J(v) + \lambda_c(1 - \int_D |v|^p)^+ + \lambda(|\Omega_u| - c)^+$$

In this section we give an existence result for the problem (8) and indicate its relationship with the problems (4) and (5). We have the following results.

Proposition 4.1 *Problem (8) admits at least one solution.*

Proof of Proposition 4.1
The proof is similar to the proof of Proposition 3.1.

Proposition 4.2 *Let u be a solution to problem (8). Then there exist* λ^* *depending only in* $N, p, |D|, c$ *such that for* $\lambda > \lambda^*$*, we have* $|\Omega_u| = c$*. If* $\lambda > \lambda^*$ *then we obtain*

(a) Any solution to (8) is a solution to (5) and (4)

(b) Any solution to problem (5) (equivalently (4)) is a solution to (8).

Proof of proposition(4.2)
Let u_0 be a solution of problem (5) (equivalently (4)) and solution u be a solution of (8).Assume that u satisfies $|\Omega_u| \leq c$ and $\int_D |u|^p = 1$. Then u solves problem (5) (equivalently (4)) and u_0 also solves (8).We obtain in this case

(i) $u \in \mathbb{V}_0$ (equivalently ($u \in \mathbb{V},$)) if we assume the condition (6)

(ii) $J_\lambda(u) = J(u)$ because $(|\Omega_u| - c)^+ = 0$ and $(1 - \int_D |u|^p)^+ = 0$.

(iii) $J_\lambda(u_0) = J(u_0)$ because $(|\Omega_{u_0}| - c)^+ = 0$ and $(1 - \int_D |u_0|^p)^+ = 0$.

We know that if u is solution of the problem (8), we have

$$J_\lambda(u) = J(u) \leq J(u_0) \leq J(v) \;\; \forall \;\; v \in \mathbb{V}_0 \;\text{(equivalently}(v \in \mathbb{V},))$$

This implies that u is a solution of problem (5) (equivalently (4)). On the other hand, as $u \in \mathbb{V}_0$ (equivalently ($u \in \mathbb{V},$)) and u_0 also solves problem (5) (equivalently (4)), we obtain

$$J_\lambda(u_0) = J(u_0) \leq J(u) = J_\lambda(u) \leq J_\lambda(v) \;\; \forall \;\; v \in W_0^{1,p}(D).$$

Finally, we have u_0 also solution of problem (8).

We show if u solution of problem (8), then we have $|\Omega_u| = c$, for λ large enough.
It is sufficient to prove that $|\Omega_u| \leq c$ for any solution to problem (8).
Let u be a solution of problem (8) and assume that $|\Omega_u| > c$. Then there exists t_0 such that for $0 < t \leq t_0$, we get $|\Omega_{u^t}| = |\Omega_u| - |\{0 < u < t\}| > c$ where $u^t = (u-t)^+$. Since u solution of the problem (8), $J_\lambda(u) \leq J_\lambda(u^t)$, we have

$$\int_{\{0 < u < t\}} |\nabla u|^p dx + \lambda |\{0 < u < t\}| \leq pt\lambda_c \int_D u^{p-1}$$

This implies that

$$\lambda \leq \frac{pt\lambda_c \int_D u^{p-1} dx}{|\{0 < u < t\}|} = \frac{pt\lambda_c ||u^{p-1}||_{L^1(D)}}{|\{0 < u < t\}|} \tag{9}$$

and

$$\int_{\{0 < u < t\}} |\nabla u|^p dx \leq pt\lambda_c ||u^{p-1}||_{L^1(D)} \tag{10}$$

Let us take $F_t := \{u > t\}$ for $0 < t \leq t_0$. For $a.e\ t \in (0, t_0)$ it holds that $\chi_{F_t} \in BV(D)$ and $D \cap \partial F_t = \{u = t\}$, for details see [19]. By the isoperimetric inequality, there exists a constant β depending only on N such that

$$|F_t|^{\frac{N-1}{N}} \leq \beta \int_D |\nabla \chi_{F_t}| = \beta P_D(F_t) \;\text{ for } a.e\ t > 0 \tag{11}$$

where $P_D(F_t)$ is the De Giorgi perimeter of F_t in D. We know that $P_D(F_t) = \mathcal{H}^{N-1}(\{u = t\}) < \infty$ for $a.e\, t \in (0, t_0)$ where $\mathcal{H}^{N-1}$ is the $(N-1)-$dimensional Hausdorff measure. But $|\Omega_{u^t}| = |F_t|$ for any $t \in (0, t_0)$. From the definition of t_0 we get $c < |F_t|$ for any $t \in (0, t_0)$.
Integrating inequality (11) on $(0, t_0)$ we have

$$c^{\frac{N-1}{N}} t_0 \leq \beta \int_0^{t_0} \mathcal{H}^{N-1}(\{u = t\}) dt$$

According to the Coarea formula see [15, 19], we have

$$c^{\frac{N-1}{N}} t_0 \leq \beta \int_{\{0 < u < t_0\}} |\nabla u|$$

If we apply the Hölder inequality, we get

$$c^{\frac{N-1}{N}} t_0 \leq \beta \left(\int_{\{0 < u < t_0\}} |\nabla u|^p \right)^{\frac{1}{p}} |\{0 < u < t_0\}|^{\frac{1}{p'}}.$$

Inequality (10) implies

$$\begin{aligned} c^{\frac{N-1}{N}} t_0 &\leq \beta \Big(p t_0 \lambda_c ||u^{p-1}||_{L^1(D)} \Big)^{\frac{1}{p}} |\{0 < u < t_0\}|^{\frac{1}{p'}} \\ c^{\frac{N-1}{N}} t_0^{\frac{1}{p'}} &\leq \beta \Big(p \lambda_c ||u^{p-1}||_{L^1(D)} \Big)^{\frac{1}{p}} |\{0 < u < t_0\}|^{\frac{1}{p'}} \end{aligned}$$

This implies that

$$\frac{t_0}{|\{0 < u < t_0\}|} \leq \left(\beta \frac{\Big(p \lambda_c ||u^{p-1}||_{L^1(D)} \Big)^{\frac{1}{p}}}{c^{\frac{N-1}{N}}} \right)^{p'}$$

Let's take

$$\lambda^* := \left(\beta \frac{\Big(p \lambda_c ||u^{p-1}||_{L^1(D)} \Big)^{\frac{1}{p}}}{c^{\frac{N-1}{N}}} \right)^{p'} p \lambda_c ||u^{p-1}||_{L^1(D)}$$

From (9) we deduce that $\lambda \leq \lambda^*$, and conclude that for $\lambda > \lambda^*$ it is impossible to have $|\Omega_u| > c$.

■

5 Hölder continuity result

By the following proposition, we derive an $L^\infty-$ estimate for solution of (8).

Proposition 5.1 *Assume that condition (7) is true. Then any solution to (8) is bounded, i.e*

$$||u||_{L^\infty(D)} \leq C$$

where C is a constant depending only on p, q, N, c and λ. We need the following Lemma which can be found in [28], we derive an $L^\infty-$ estimate for solution of (8).

To prove this proposition, we need the following lemma which can be found in [28], lemma 5.2 page 71.

Lemma 5.1 *If the function u belongs to $W^{1,p}(D)$, $1 < p \leq N$, such that $essmax_{\partial D} u < \infty$. For $k \geq k_0 \geq essmax_{\partial D} u$, set $A_k := \{u > k\}$. If u satisfies the inequalities*

$$\int_{A_k} |\nabla u|^p dx \leq \gamma k^\eta |A_k|^{(1-\frac{p}{N}+\epsilon)} \tag{12}$$

where the constants γ, $\epsilon > 0$ and $0 \leq \eta \leq \epsilon + p$. Then there exists a constant C depending only in $\gamma, \eta, p, N, \epsilon, k_0$ and $||u||_{L^1(A_{k_0})}$ such that essential $max_D u$ is bounded by C.

Remark 5.1 *In the proof of Lemma 5.3 presented in [28], one can see that $||u||_{L^1(A_{k_0})} \leq K$ for some constant K then u is bounded by a constant C which depends only in $\gamma, \eta, p, N, \epsilon, k_0$ and K.*

Proof of Proposition 5.1

We prove that u satisfies an inequality like (12). Since $u \in W^{1,p}(D)$, we have $essmax_{\partial D} u = 0$. So $k_0 = 0$ and for $k > 0$ set $A_k = \{u > k\}$ and we consider the function

$$u_k = \begin{cases} k & \text{if} \quad u > k \\ u & \text{if} \quad u \leq k, \end{cases}$$

Since $|\Omega_u| = |\Omega_{u_k}|$ for any $k \geq k_0$ and by minimality of u we have

$$\int_{A_k} |\nabla u|^p dx \quad \leq \quad p\lambda_c \int_{A_k} |u|^{p-1}(u-k)$$

By the Hölder inequality we obtain

$$\int_{A_k} |\nabla u|^p dx \quad \leq \quad p\lambda_c ||u^{p-1}||_{L^q(A_k)} ||(u-k)^+||_{L^{\frac{q}{q-1}}(A_k)}$$

According by inequality (5) for $s = \frac{q}{q-1}$ implies that

$$\int_{A_k} |\nabla u|^p dx \quad \leq \quad p\lambda_c ||u^{p-1}||_{L^q(A_k)} |A_k|^{\frac{q-1}{q} - \frac{N-p}{Np}} ||\nabla u||_{L^p(A_k)}$$

and finally we obtain

$$\int_{A_k} |\nabla u|^p dx \quad \leq \quad (p\lambda_c ||u^{p-1}||_{L^q(A_k)})^{\frac{p}{p-1}} |A_k|^{\frac{p}{p-1}(\frac{q-1}{q} - \frac{N-p}{Np})}$$

We found here inequality (12) with $\eta = 0$, $\epsilon = \frac{p(pq-N)}{q(p-1)N}$ and $\gamma = (p\lambda_c ||u^{p-1}||_{L^q(A_k)})^{\frac{p}{p-1}}$. ■

Theorem 5.1 *Let u be a solution to (8) and assume that (7). Then u is locally Hölder continuous in D.*

Proof of Theorem 5.1
Let B_r be a ball with radius r such that $\bar{B}_r \subset D$ and $\mu \in W^{1,\infty}(D)$ such that $0 \leq \mu \leq 1$ in D and $\mu = 0$ on $B_r^c = \bar{D} \backslash B_r$. For $k \in \mathbb{R}$ we set $v := u - (u-k)^+ \mu^p$ and $A_{k,r} := B_r \cap \{u > k\}$.

In $A_{k,r}^c$ we get $v = u$. Let 's take $\Omega_v = \Omega_u \cap A_{k,r}^c \cup (\Omega_v \cap A_{k,r})$, we see that

$$(|\Omega_v| - c)^+ \leq (|\Omega_v| - c)^+ + |A_{k,r}|$$

Combining this inequality with the relation $J_\lambda(u) \leq J_\lambda(v)$, we obtain

$$\begin{aligned} \int_{A_{k,r}} |\nabla u|^p dx &\leq \int_{A_{k,r}} |\nabla v|^p dx \\ &+ p\lambda_c \int_{A_{k,r}} |u|^{p-1}(u-k)\mu^p dx + \lambda p |A_{k,r}| \end{aligned} \tag{13}$$

Writing $|\nabla v| \leq (1-\mu)^p |\nabla u| + \mu^p \frac{p(u-k)}{\mu} |\nabla \mu|$ in $A_{k,r} \cap \{\mu \neq 0\}$ and using the convexity of the function $x \in (0,\infty) \longrightarrow x^p$, we obtain

$$\begin{aligned} \int_{A_{k,r}} |\nabla v|^p &\leq \int_{A_{k,r}} |\nabla u|^p dx - \int_{A_{k,r}} \mu^p |\nabla u|^p dx \\ &+ p^p \int_{A_{k,r}} (u-k)^p |\nabla \mu|^p dx \end{aligned} \tag{14}$$

On the other hand, thanks to Young's inequality, we have

$$(u-k)\mu^p \leq \frac{1}{p}((u-k)\mu^p)^p + \frac{1}{p'} \text{ where } p' = \frac{1}{p-1}.$$

Since the condition (7) holds we obtain by the Hölder inequality

$$\begin{aligned} p\lambda_c \int_{A_{k,r}} |u|^{p-1}(u-k)\mu^p dx &\leq + (p-1)|A_{k,r}|^{\frac{q-1}{q}} \Big) \\ & \lambda_c ||u^{p-1}||_{L^q} \Big(\Big(\int_{A_{k,r}} ((u-k)\mu^p)^{\frac{q}{q-1}} \Big)^{\frac{q-1}{q}} \end{aligned} \tag{15}$$

According to inequality (5) we take $s = \frac{q}{q-1}$, we obtain

$$\Big(\int_{A_{k,r}} ((u-k)\mu^p)^{\frac{q}{q-1}} \Big)^{\frac{q-1}{q}} \leq C|A_{k,r}|^{1+\frac{1}{N}-\frac{1}{p}-\frac{1}{q}} \int_{A_{k,r}} |\nabla((u-k)\mu^p)|^p \tag{16}$$

But on $A_{k,r}$, we have $|\nabla((u-k)\mu^p)| \leq \mu^{p-1}(p(u-k))|\nabla \mu| + \mu |\nabla u|$. Using the convexity of the function $x \in (0,\infty) \longrightarrow x^p$, we obtain

$$|\nabla((u-k)\mu^p)|^p \leq 2^{p-1}\mu^{p(p-1)}(p^p(u-k)^p|\nabla \mu|^p + \mu^p |\nabla u|^p) \tag{17}$$

Passing to integral we get

$$\int_{A_{k,r}} |\nabla((u-k)\mu^p)|^p \le 2^{p-1}(p^p \int_{A_{k,r}} \mu^{p(p-1)}(u-k)^p|\nabla\mu|^p + \int_{A_{k,r}} \mu^p|\nabla u|^p) \tag{18}$$

Inequality (18), together with (15) and (16) imply that

$$\begin{cases} p\lambda_c \int_{A_{k,r}} |u|^{p-1}(u-k)\mu^p dx \le \lambda_c ||u^{p-1}||_{L^q}\Big(2^{p-1}C^p|A_{k,r}|^{1+\frac{1}{N}-\frac{1}{p}-\frac{1}{q}} \\ \Big(p^p \int_{A_{k,r}} \mu^p|\nabla u|^p + \\ p^p \int_{A_{k,r}} \mu^{p(p-1)}(u-k)^p|\nabla\mu|^p\Big) \quad +(p-1)|A_{k,r}|^{1-\frac{1}{q}}\Big) \end{cases} \tag{19}$$

For $r \le r_0$ where

$$r_0 = (2^pC^p w_N^{1+\frac{1}{N}-\frac{1}{p}-\frac{1}{q}} \lambda_c ||u^{p-1}||_{L^q(D)}), \text{ with } w_N = |B(0,1)|$$

we have

$$\lambda_c ||u^{p-1}||_{L^q(D)}(2^{p-1}C|A_{k,r}|^{1+\frac{1}{N}-\frac{1}{p}-\frac{1}{q}}) \le \frac{1}{2}$$

and

$$p\lambda_c \int_{A_{k,r}} |u|^{p-1}(u-k)\mu^p dx \le \frac{1}{2}\int_{A_{k,r}} \mu^p|\nabla u|^p + \frac{p^p}{2}\int_{A_{k,r}} \mu^{p(p-1)}(u-k)^p|\nabla\mu|^p + \Big((p-1)\lambda_c||u^{p-1}||_{L^q(D)}|A_{k,r}|^{\frac{1}{q}}\Big)|A_{k,r}|^{1-\frac{1}{q}} \tag{20}$$

Inequality (20) combining with inequality (13) and (14), imply that for $r \le r_0$,

$$\int_{A_{k,r}} \mu^p|\nabla u|^p \le 3p^p \int_{A_{k,r}} |\nabla\mu^p|(u-k)^p + ((p-1)\lambda_c||u^{p-1}||_{L^q(D)} + p\lambda|A_{k,r}|^{\frac{1}{q}})|A_{k,r}|^{1-\frac{1}{q}}$$

We use the fact that $0 \le \mu \le 1$. For $r \le r_0$, we get $|A_{k,r}|^{\frac{1}{q}} \le |B_{r_0}|^{\frac{1}{q}} \le C$ where

$$C = w_N(2^pC^p w_N^{1+\frac{1}{N}-\frac{1}{p}-\frac{1}{q}} \lambda_c ||u^{p-1}||_{L^q(D)})^{\frac{-N}{q-N}}$$

We obtain

$$\int_{A_{k,r}} \mu^p|\nabla u|^p \le \gamma\Big(\int_{A_{k,r}} |\nabla\mu^p|(u-k)^p + |A_{k,r}|^{1-\frac{1}{q}}\Big) \tag{21}$$

where $\gamma = max\{3p^p, (p-1)\lambda_c||u^{p-1}||_{L^q(D)} + pC_1\lambda\}$. We see that u belongs to the generalized De Giorgi class $\mathcal{B}_p(D, ||u||_{L^\infty(D)}, \gamma, +\infty, \frac{1}{q})$. Considering the theorem 6.1 of chapter 2 in [28], we conclude u is locally Hölder continuous in D. ■

Acknowledgements *The author thanks for a very pleasant visit to ICTP, Italy, Trieste as associate.*

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