AMO - Advanced Modeling and Optimization, Volume 14, Number 3, 2012

Transportation inequalities for SDEs involving fractional Brownian motion and standard Brownian motion

Toufik Guendouzi [1]

Laboratory of Stochastic Models, Statistical and Applications
Tahar Moulay University PO.Box 138 En-Nasr, 20000 Saida, Algeria
e-mail: tf.guendouzi@gmail.com

Abstract

For stochastic differential equations driven by fractional Brownian motion and standard Brownian motion, we prove that $W_1\mathbb{H}$ transportation inequalities hold for the law of its solution on the space of $\mathbb{R}^d$-valued continuous function on $[0,T]$ w.r.t the L^1-metric, under some assumptions on the coefficients, via Malliavin calculus.

Key words and phrases: Stochastic differential equation, Transportation inequality, Fractional Brownian motion, Malliavin calculus.
2000 MR Subject Classifications: 60H10, 28A35, 60F15, 60J60, 60G05.

1 Introduction

Let $B^H = \{B_t^H, t \in \mathbb{R}_+\}$ be an d-dimensional fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$, i.e. a zero mean Gaussian process with covariance function $\mathbb{E}(B_s^i, B_t^j) = R_H(s,t)\delta_{ij}$, where

$$R_H(s,t) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t-s|^{2H}\Big). \tag{1}$$

This process is a ordinary Brownian motion when $H = \frac{1}{2}$. From (1) we deduce that $\mathbb{E}\Big(\Big|B_t^H, B_s^H\Big|^2\Big) = d|t-s|^H$ and, as a consequence, the trajectories of B^H are almost surely locally α-Hölder continuous for all $\alpha \in (0,H)$. We refer the reader to [1, 10] and references therein for further information about fBm and stochastic integration with respect to this process.

[1]The work of the corresponding author is supported by The National Agency of Development of University Research (ANDRU), Algeria (SMA 2011-2014).
***AMO-Advanced Modeling and Optimization. ISSN: 1841-4311**

In this article we fix a time interval $[0,T]$ and $\frac{1}{2} < H < 1$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and we consider the following mixed stochastic differential equation

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma_W(s,X_s)dW_s + \int_0^t \sigma_H(s,X_s)dB_s^H, \qquad (2)$$

where $X_0 \in \mathbb{R}^d$ is the initial value of the process X, W is an n-dimensional standard Brownian motion and B^H is an m-dimensional fractional Brownian motion with $H \in (\frac{1}{2}, 1)$. The coefficients $b : [0,t] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma_W : [0,t] \times \mathbb{R}^d \to \mathbb{R}^{d\times n}$ are assumed Lipschitz continuous in x and $\sigma_H : [0,t] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ has bounded partial derivatives which are Hölder continuous of order $\lambda > \frac{1}{H} - 1$, and the processes W and B^H are independent.

For each $t \in [0,T]$ we denote by $\mathcal{F}_t$ the σ-filed generated by the random variables $\{X_0, B_s^H, W_s, s \in [0,t]\}$ and the $\mathbb{P}$-null sets. The integral $\int_0^t \sigma_W(s,X_s)dW_s$ in the SDE (2) should be interpreted as an Itô stochastic integral and $\int_0^t \sigma_H(s,X_s)dB_s^H$ as a pathwise Riemann-Stieltjes integral, which can be expressed as a Lebesgue integral using a fractional integration by parts formula (see [17]).

For several years, the transportation cost-information inequalities and thier applications to diffusion processus have been studies (see [2], [5]). In a recent paper, Liming Wu [16] established the transportation inequalities for stochastic differential equations of pure jumps. In that paper, it was shown that for stochastic differential equations of pure jumps the $W_1\mathbb{H}$ transportation inequalities hold for its invariant probability measure and for its process-level law on right continuous paths space under the dissipative condition. However, the results in that paper was applied to concentration inequalities. The main purpose of this work is to apply the recent results on the mixed stochastic differential equations involving fractional Brownian motion and standard Brownian motion in order to obtain transportation inequalities.

Let (E,d) denote the metric space equipped with σ-field such that the distance d is $\mathcal{B}\otimes\mathcal{B}$-measurable. For any $p \geq 1$ and two probability measures μ and ν on E, recall the L^p-Wasserstein distance between μ and ν

$$W_p^d(\mu,\nu) = \inf\Big(\int\int d(x,y)^p d\pi(x,y)\Big)^{\frac{1}{p}}.$$

This quantity is finite as soon as μ and ν have finite moments of order p. The Kullback information of ν with respect to μ is given by

$$\mathbb{H}(\nu/\mu) = \begin{cases} \int \log \frac{d\nu}{d\mu} d\nu & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

We say that the probability measure μ satisfies the L^p-transportation inequality on (E,d) ($\mu \in T_p(C)$ for short) if there existe a constant $C \geq 0$ such that for any probability measure ν,

$$W_p^d(\mu,\nu) \leq \Big(2C\mathbb{H}(\nu/\mu)\Big)^{\frac{1}{2}}. \tag{3}$$

Let ϕ be a increasing function and left-continuous on $\mathbb{R}_+$ which vanishes at 0. The probability measure μ satisfies a $W_1\mathbb{H}$ inequality with deviation function ϕ if

$$\phi\Big(W_{1,d}(\mu,\nu)\Big) \leq \mathbb{H}(\nu/\mu), \quad \forall \nu \in \mathcal{P}_d(E), \tag{4}$$

where $\mathcal{P}_d(E)$ is the set of all probability measures ν such that $\int d(x_0,x)\nu(dx) < \infty$. The inequality (3) is a particular case with $\phi(t) = t^{2/p}/(2C), \quad t \geq 0$.

By means of Malliavin calculus, Wu in [16] proved that the distribution $P_T(x,dy)$ of $X_t(x)$ satisfies the $W_1\mathbb{H}$ transportation inequality and $\mathbb{P}_{x,[0,T]}$ satisfies on the space $D([0,T],\mathbb{R}^d)$ of right continuous left limits $\mathbb{R}^d$-valued functions on $[0,T]$, for some deviation function ϕ_T

$$\phi\Big(W_{1,d_{L^1}}(Q,\mathbb{P}_x)\Big) \leq \mathbb{H}(Q/\mathbb{P}_x), \quad Q \in M_1(D([0,T],\mathbb{R}^d)).$$

The structure of the paper is as follows: in the next section we recall some classical definitions and results on fractional calculus and we list our assumptions on the coefficients of Eq. (2). In section 3 we state the main results of our paper. Section 4 is devoted to relating the Malliavin calculus to our case and to prove Theorem 3.2 and Theorem 3.3 enounced in section 3.

2 Preliminaries

Let $d,l \in \mathbb{N}^*$. Given the matrix $A = (a^{i,j})_{d\times l}$ and a vector $y = (y^i)_{d\times 1}$ we denote $|A|^2 = \sum_{i,j} |a^{i,j}|^2$ and $|y|^2 = \sum_i |y^i|^2$.

Let $\alpha \in (0,\frac{1}{2})$. For any measurable function $f : [0,t] \to \mathbb{R}^d$ we introduce the following notation

$$||f(t)||_\alpha := |f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}}ds. \tag{5}$$

Denote by $W_0^{\alpha,\infty}$ the space of measurable functions $f : [0,t] \to \mathbb{R}^d$ such that

$$||f(t)||_{\alpha,\infty} := \sup_{t\in[0,T]} ||f(t)||_\alpha < \infty. \tag{6}$$

A equivalent norm can be defined by

$$||f||_{\alpha,\varrho} = \sup_{t\in[0,T]} e^{-\varrho t}\left(|f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}}ds\right); \quad \varrho \geq 0. \tag{7}$$

For $0 < \lambda \leq 1$, denote by $C^\lambda(0,T,\mathbb{R}^d)$ the space of λ-Hölder continuous functions $f : [0,T] \to \mathbb{R}^d$, equipped with the norm

$$||f||_\lambda := ||f||_\infty + \sup_{0\leq s<t\leq T} \frac{|f(t)-f(s)|}{(t-s)^\lambda} < \infty, \tag{8}$$

where $||f(t)||_\infty := \sup_{t\in[0,T]} |f(t)|$. Note that for any ϵ, $(0 < \epsilon < \alpha)$, we have the inclusions

$$C^{\alpha+\epsilon}([0,T];\mathbb{R}^d) \subset W_0^{\alpha,\infty}([0,T];\mathbb{R}^d) \subset C^{\alpha-\epsilon}([0,T];\mathbb{R}^d).$$

We denote by $W_T^{1-\alpha,\infty}([0,T];\mathbb{R}^d)$ the space of continuous functions $g:[0,T]\to\mathbb{R}^d$ such that

$$||g||_{1-\alpha,\infty,T} := \sup_{0<s<t<T}\left(\frac{|g(t)-g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y)-g(s)|}{(y-s)^{2-\alpha}}dy\right) < \infty.$$

Clearly, for all $\epsilon > 0$ we have

$$C^{1-\alpha+\epsilon}([0,T];\mathbb{R}^d) \subset W_T^{1-\alpha,\infty}([0,T];\mathbb{R}^d) \subset C^{1-\alpha}([0,T];\mathbb{R}^d).$$

Denoting

$$\Lambda_\alpha(g;[0,T]) = \frac{1}{\Gamma(1-\alpha)}\sup_{0<s<t<T} |(D_{t^-}^{1-\alpha} g_{t^-})(s)|,$$

where $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1}e^{-r}dr$ is the Euler function and

$$(D_{t^-}^{1-\alpha} g_{t^-})(s) = \frac{e^{i\pi(1-\alpha)}}{\Gamma(\alpha)}\left(\frac{g(s)-g(t)}{(t-s)^{1-\alpha}} + (1-\alpha)\int_s^t \frac{g(s)-g(y)}{(y-s)^{2-\alpha}}dy\right)\mathbf{1}_{(0,t)}(s).$$

We also define the space $W_1^\alpha(0,T,\mathbb{R}^d)$ of measurable functions f on $[0,T]$ such that

$$||f||_{\alpha,1;[0,T]} = \int_0^T \left[\frac{|f(t)|}{t^\alpha} + \int_0^t \frac{|f(t)-f(y)|}{(t-y)^{\alpha+1}}dy\right]dt < \infty.$$

We have $W_\infty^\alpha(0,T,\mathbb{R}^d) \subset W_1^\alpha(0,T,\mathbb{R}^d)$ and $||f||_{\alpha,1;[0,T]} \le \left(T + \frac{T^{1-\alpha}}{1-\alpha}\right)||f||_{\alpha,\infty;[0,T]}$.

In [17] Zähle introduced the generalized Stieltjes integral

$$\int_0^T f(t)dg(t) = (-1)^\alpha \int_0^T (D_{0+}^\alpha f)(t)(D_{T-}^{1-\alpha} g_{T-})(t)dt, \tag{9}$$

defined in terms of the fractional derivative operators

$$(D_{0+}^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(t)}{t^\alpha} + \alpha\int_0^t \frac{f(t)-f(y)}{(t-y)^{\alpha+1}}dy\right)\mathbf{1}_{(0,T)}(t),$$

and

$$(D_{T-}^\alpha g_{T-})(t) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)}\left(\frac{g(t)-g(T)}{(T-t)^\alpha} + \alpha\int_t^T \frac{g(t)-g(y)}{(t-y)^{\alpha+1}}dy\right)\mathbf{1}_{(0,T)}(t).$$

We refer the reader to [10] for further details on fractional operators.

the following propositions are the estimates of the generalized Stieltjes integral.

Proposition 2.1 ([10]) *Fix $0 < \alpha < \frac{1}{2}$. Given two functions $g \in W_T^{1-\alpha,\infty}(0,T)$ and $f \in W_0^{\alpha,1}(0,T)$ we set*

$$G_s^t(f) = \int_s^t f_r dg_r.$$

Then for all $r < t \leq T$ we have

$$\begin{aligned} \left| \int_s^t f_r dg_r \right| &\leq \sup_{s \leq r < \tau \leq t} |(D_{\tau-}^{1-\alpha} g_{\tau-})(r)| \int_s^t |(D_{s+}^{\alpha} f)(\tau)| d\tau \\ &\leq \Lambda_\alpha(g; [s,t]) ||f||_{\alpha,1;[0,T]}. \end{aligned} \tag{10}$$

Proposition 2.2 *Let $0 < \alpha < \frac{1}{2}$.*

1. *$G_{s,t} : W^{\alpha,\infty}(s,t;\mathbb{R}^d) \to C^{1-\alpha}([s,t];\mathbb{R}^d)$ is a linear continuous map and*

$$||G_{s,.}(f)||_{1-\alpha;[s,t]} \leq c_{\alpha,t}^{(1)} \Lambda_\alpha(g;[s,t]) ||f||_{\alpha,\infty;[0,T]}, \tag{11}$$

where $c_{\alpha,t}^{(1)}$ is a positive constant depending only on α, t and $c_{\alpha,t}^{(1)} \leq 4 + 3t$.

2. *$G_{s,t} : W^{\alpha,\infty}(s,t;\mathbb{R}^d) \to G_{s,t} : W^{\alpha,\infty}(s,t;\mathbb{R}^d)$ is a linear continuous map and for $\forall \lambda > 1$*

$$||G_{s,.}(f)||_{\alpha,\lambda;[s,t]} \leq \frac{\Lambda_\alpha(g;[s,t])}{\lambda^{1-2\alpha}} c_{\alpha,t}^{(2)} ||f||_{\alpha,\lambda;[s,t]}, \tag{12}$$

$c_{\alpha,t}^{(2)}$ is a positive constant depending only on α, t and $c_{\alpha,t}^{(2)} = \frac{4}{1-2\alpha}\left(\frac{2}{\alpha} + t^\alpha\right)$.

Proof.

1. Let $s \leq r < \tau \leq t$. From (10) we have

$$|G_{r,\tau}(f)| \leq \Lambda_\alpha(g;[s,t]) \int_r^\tau \left(\frac{|f(\theta)|}{(\theta - r)^\alpha} + \alpha \int_r^\theta \frac{|f(\theta) - f(u)|}{(\theta - u)^{\alpha+1}} du \right) d\theta \tag{13}$$

and therefore

$$|G_{r,\tau}(f)| \leq \Lambda_\alpha(g;[s,t])(2 + t^\alpha)(\tau - r)^{1-\alpha} ||f||_{\alpha,\infty;[s,t]}. \tag{14}$$

The inequality (11) follows with $c_{\alpha,t}^{(1)} = \frac{t^{1-\alpha}}{1-\alpha} + t + 2 + t^\alpha \leq 4 + 3t$.

2. We have

$$\begin{aligned} &\int_s^\tau \frac{|G_{s,\tau}(f) - G_{s,r}(f)|}{(\tau - r)^{1+\alpha}} dr \\ &\leq \Lambda_\alpha(g;[s,t]) \int_s^\tau (\tau - r)^{-\alpha-1} \times \left(\int_r^\tau \frac{|f(\theta)|}{(\theta - r)^\alpha} + \alpha \int_r^\theta \frac{|f(\theta) - f(u)|}{(\theta - u)^{\alpha+1}} du d\theta \right) dr \\ &\leq \Lambda_\alpha(g;[s,t]) \int_s^\tau |f(\theta)| \int_s^\theta (\tau - r)^{-\alpha-1} (\theta - r)^{-\alpha} dr d\theta + \\ &+ \Lambda_\alpha(g;[s,t]) \int_s^\tau \int_s^\theta \frac{|f(\theta) - f(u)|}{(\theta - u)^{\alpha+1}} \left(\int_s^u (\tau - r)^{-\alpha-1} dr \right) du d\theta. \end{aligned}$$

From

$$\begin{aligned}
&\int_s^\theta (\tau - r)^{-\alpha-1}(\theta - r)^{-\alpha} dr \\
&\quad = (\tau-\theta)^{-2\alpha} \int_0^{\frac{\theta-s}{\tau-\theta}} (1+u)^{-\alpha-1} u^{-\alpha} du \\
&\quad \leq (\tau-\theta)^{-2\alpha} \left(\int_0^1 (1+u)^{-\alpha-1} u^{-\alpha} du + \int_1^\infty (1+u)^{-\alpha-1} u^{-\alpha} du \right) \\
&\quad \leq \left(\tfrac{1}{1-\alpha} + \tfrac{1}{\alpha} \right) (\tau-\theta)^{-2\alpha},
\end{aligned}$$

and for $s < r < u < \theta < \tau$

$$\int_s^u (s-r)^{-\alpha-1} dr \leq \frac{1}{\alpha}(\tau-u)^{-\alpha} \leq \frac{t^\alpha}{\alpha}(\tau-s)^{-2\alpha},$$

we have $|G_{s,\tau}(f)| + \int_s^\tau \frac{|G_{s,\tau}(f) - G_{s,r}(f)|}{(\tau-r)^{1+\alpha}} dr$

$$\begin{aligned}
&\leq \Lambda_\alpha(g;[s,t]) \left(\tfrac{1}{(1-\alpha)\alpha} + t^\alpha \right) \\
&+ \int_s^\tau ((\tau-\theta)^{-2\alpha} + (\theta-s)^{-\alpha}) \left(|f(\theta)| + \int_s^\theta \frac{|f(\theta) - f(u)|}{(\theta-u)^{\alpha+1}} du \right) d\theta.
\end{aligned}$$

Finally, the inequality (12) follows sine

$$\begin{aligned}
&\int_s^\tau e^{-\lambda(\tau-\theta)} [(\tau-\theta)^{-2\alpha} + (\theta-s)^{-\alpha}] d\theta \\
&\quad = \tfrac{1}{\lambda^{1-2\alpha}} \int_0^{\lambda(\tau-s)} e^{-u} u^{-2\alpha} du + \frac{1}{\lambda^{1-\alpha}} e^{-\lambda(\tau-s)} \int_0^{\lambda(\tau-s)} e^u u^{-\alpha} du \\
&\quad \leq \tfrac{1}{\lambda^{1-2\alpha}} \tfrac{4}{1-2\alpha}.
\end{aligned}$$

□

In our article we are interested in the solution of the equation on $\mathbb{R}^d$

$$X_t^i = X_0^i + \int_0^t b^i(s, X_s) ds + \sum_{k=1}^n \int_0^t \sigma_W^{i,k}(s, X_s) dW_s^k + \sum_{j=1}^m \int_0^t \sigma_H^{i,j}(s, X_s) dB_s^{H,j}, \quad (15)$$

we consider the following assumptions on the coefficients, which are supposed to hold for $\mathbb{P}$-almost all $\omega \in \Omega$: There exist some constants $\beta, \delta, 0 < \beta, \delta \leq 1$, and for every $K \geq 0$ the following properties hold

($\mathbf{H}_1$) Set $\alpha_0 = \min\left\{\frac{1}{2}, \beta, \frac{\delta}{1+\delta}\right\}$, there exists $c_K > 0$, $\vartheta \in (1-\alpha_0, 1]$ such that for all $t \in [0,T]$ and all $x, y \in \mathbb{R}^d$

$$|b(t,x) - b(s,y)| \leq c_K(|t-s|^\vartheta + |x-y|), \qquad \forall |x|, |y| \leq K;$$

($\mathbf{H}_2$) There exists the function $b_0 \in c_N(0,T;\mathbb{R}^d)$ with $N \geq 2$ and $c > 0$ such that

$$\sup_{s\in[0,T]} \sup_{x\in\mathbb{R}^d} |b(s,x)| \leq c|x| + b_0, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^d;$$

($\mathbf{H}_3$) There exists $\bar{c} \in \mathbb{R}$ such that for any $x, y \in \mathbb{R}^d$

$$\langle x-y, b(t,x) - b(s,y) \rangle_{\mathbb{R}^d} \leq \bar{c}(|t-s|^\vartheta + |x-y|^2);$$

$(\mathbf{H}_4)$ There exists $1-H < \alpha < \alpha_0 = \min\left\{\frac{1}{2}, \beta, \frac{\delta}{1+\delta}\right\}$ such that $\sigma_H \in C^{\alpha}(0,T;\mathbb{R}^{d\times m})$;

$(\mathbf{H}_5)$ There exists $\tilde{c}_0 > 0$ such that for all $t \in [0,T]$

$$|\sigma_H(t,x) - \sigma_H(s,y)| \leq \tilde{c}_0(|t-s|^{\beta} + |x-y|), \qquad \forall x, y \in \mathbb{R}^d;$$

$(\mathbf{H}_6)$ There exists $\hat{c}_0 > 0$, $\check{c}_0 > 0$ and $0 < \eta < \min\left\{\frac{1}{2}, \beta, \frac{\delta}{2}\right\}$ such that $\sigma_W \in C^{\eta}(0,T;\mathbb{R}^{d\times n})$ and for all $t \in [0,T]$, $\sup_{s\in[0,T]} \sup_{x\in\mathbb{R}^d} |\sigma_W(s,x)| \leq \check{c}_0(1+|x|^{\eta})$,

$$|\sigma_W(t,x) - \sigma_W(s,y)| \leq \hat{c}_0(|t-s|^{\eta} + |x-y|), \qquad \forall x, y \in \mathbb{R}^d.$$

3 Mains results

Let us start by recalling the following result, which is a characterization for the $W_1\mathbb{H}$ transportation inequality. This result has been proved in [6].

Let μ be a fixed probability measure on the metric space (E, d).

Lemma 3.1 ([6]) *Let $\phi : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ be a non-decreasing left-continuous convex function with $\phi(0) = 0$. The following properties are equivalent:*

1. *the $W_1\mathbb{H}$ inequality below holds:*
$$\phi(W_{1,d}(\nu,\mu)) \leq \mathbb{H}(\nu/\mu), \tag{16}$$
where ν is a measure on the metric space E;

2. *for every $F : (E,d) \to \mathbb{R}$ bounded and Lipschitzian with $||F||_{Lip} \leq 1$,*
$$\int \exp(\zeta(F - \mu(F)))d\mu \leq \exp(\hat{\phi}(\zeta)), \qquad \zeta > 0, \tag{17}$$
where $\hat{\phi}(\zeta) = \sup_{r\geq 0}(r\zeta - \phi(r))$ is the semi-Legendre transformation;

3. *for every $F : (E,d) \to \mathbb{R}$ with $||F||_{Lip} \leq 1$,*
$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n} F(\xi_k) - \mu(F) > r\right) \leq \exp(-n\phi(r)), \quad r > 0, n \geq 1, \tag{18}$$
where $(\xi_k)_{k\geq 1}$ is a sequence of i.i.d.r.v. valued in E of common law μ.

In such case ϕ is called a $W_1\mathbb{H}$-deviation function of μ.

Theorem 3.2 *Let $\mathbb{P}_x$ be the law of solution of the stochastic differential equation* (15)*, and assume that the coefficients b, σ_H and σ_W satisfy the assumptions* $(\mathbf{H}_1)$-$(\mathbf{H}_2)$-$(\mathbf{H}_3)$, $(\mathbf{H}_4)$-$(\mathbf{H}_5)$ *and* $(\mathbf{H}_6)$. *Then there exists a positive constant $\widetilde{K}$ depending only on $c_K, c, \bar{c}, \tilde{c}_0, \hat{c}_0, \check{c}_0$ and T such that the probability measure $\mathbb{P}_x$ satisfies the L^1-transportation inequality on the metric space $C([0,T];\mathbb{R}^d)$ equipped with the metric*

$$d_{L^1}(\gamma_1,\gamma_2) = \int_0^T |\gamma_1(t) - \gamma_2(t)|dt,$$

with $C := C_{T,H} = \dfrac{2}{\widetilde{K}^2} H T^{2H} ||\sigma_H||^2_{\infty;[0,T]} c_{\widetilde{K},T} + 4T^2 ||\sigma_W||^2_{\infty;[0,T]}$ *and*

$$c_{\widetilde{K},T} := 1 - \exp(-\widetilde{K}T).$$

Now we present our mains results.

Theorem 3.3 *Let $P_t(x, dy)$ be the distribution of $X_t(x)$ solution of the SDE (15). Assume that $(\mathbf{H}_1)$-$(\mathbf{H}_2)$-$(\mathbf{H}_3)$, $(\mathbf{H}_4)$-$(\mathbf{H}_5)$ and $(\mathbf{H}_6)$ are satisfies, and there exists a positive constant $\widetilde{K}$ depending only on $c_K, c, \bar{c}, \tilde{c}_0, \hat{c}_0, \check{c}_0$ and T.*

Suppose that there exists $\tilde{\sigma} \in C^\eta([0,t])$ and $\zeta > 0$ such that for all $t \in [0,T]$ and all $x \in \mathbb{R}^d$, $|\tilde{\sigma}_t(x)| \leq ||\sigma_W(t,x)||_{0,T,\eta}$ and

$$C(\zeta) = \mathbb{E}\left[\exp\left(\zeta T^{2\eta} ||\sigma_W(.,x(.))||^2_{0,T,\eta}\right) - \zeta T^{2\eta} ||\sigma_W(.,x(.))||^2_{0,T,\eta} - 1\right] < \infty. \quad (19)$$

The following properties hold true

1. *(X_t) admits a unique invariant probability measure μ, and for all ν on $\mathbb{R}^d$*

$$W_{1,d}(\nu P_t, \mu) \leq \exp(-\widetilde{K}\eta T^{2\eta} t) W_{1,d}(\nu, \mu), \quad \forall t > 0, \quad (20)$$

   where (P_t) is the transition kernel semigroup of the Markov process (X_t), and d is the Euclidean metric .

2. *For each $T > 0$, the distribution $P_T(x, dy)$ of the solution $X_t(x)$ satisfies the following $W_1\mathbb{H}$ transportation inequality*

$$\phi_T\left(W_{1,d}(\nu, P_T(x,dy))\right) \leq \mathbb{H}(\nu / P_T(x,dy)), \quad (21)$$

   where ν is a $\mathbb{R}^d$-valued probability measure and

$$\begin{aligned} \phi_T(u) &:= \sup_{\zeta \geq 0}\left\{ u\zeta T^{2\eta} - \int_0^T C(e^{-2\widetilde{K}T^{2\eta}t} T^{2\eta}\zeta)dt - \frac{||\sigma_H(.,x(.))||^2_{0,T,\eta} T^{2-2\eta}\zeta^2}{4\widetilde{K}}(1 - e^{-2\widetilde{K}\eta T^{2\eta}t})\right\} \\ &\geq \frac{1}{\widetilde{K}} \sup_{\zeta \geq 0}\left(u\widetilde{K}\zeta T^{2\eta} - (C(\zeta T^{2\eta}) + \vartheta\zeta^2 T^{2-2\eta} ||\sigma_H(.,x(.))||^2_{0,T,\eta}/2)\right), \qquad u \geq 0. \end{aligned}$$

   In particular, for all $\mathbb{R}^d$-valued probability measure ν, set

$$C^*_\vartheta(\widetilde{K}u) = \sup_{\zeta \geq 0}\left(u\widetilde{K}\zeta T^{2\eta} - (C(\zeta T^{2\eta}) + \vartheta\zeta^2 T^{2-2\eta} ||\sigma_H(.,x(.))||^2_{0,T,\eta}/2)\right)$$

   we have

$$\frac{1}{\widetilde{K}} C^*_\vartheta(\widetilde{K} W_{1,d}(\nu, \mu)) \leq \phi_\infty(W_{1,d}(\nu,\mu)) \leq \mathbb{H}(\nu/\mu). \quad (22)$$

3. *For each $T > 0$, $\mathbb{P}_x$ satisfies on the space $C([0,T]; \mathbb{R}^d)$ of $\mathbb{R}^d$-valued continuous functions on $[0,T]$ equipped with the L^1-metric*

$$\phi^P_T(W_{1,d_{L^1}}(Q, \mathbb{P}_x)) \leq \mathbb{H}(Q|\mathbb{P}_x), \quad (23)$$

for all probability measure Q on $C([0,T];\mathbb{R}^d)$, and

$$\begin{aligned}
\phi_T^P(u) &:= \sup_{\zeta\geq 0}\left\{u\zeta T^{2\eta} - \int_0^T C\left((1-e^{-2\widetilde{K}T^{2\eta}t})T^{2\eta}\zeta/\widetilde{K}\right)dt\right.\\
&- \left.\frac{||\sigma_H(.,x(.))||^2_{0,T,\eta}T^{2-2\eta}\zeta^2}{4}\int_0^T[(1-e^{-2\widetilde{K}\eta T^{2\eta}t})/\widetilde{K}]^2dt\right\}\\
&\geq \sup_{\zeta\geq 0}\left\{u\zeta T^{2\eta} - T\left(C(\zeta T^{2\eta}/\widetilde{K}) + \zeta^2 T^{2-2\eta}||\sigma_H(.,x(.))||^2_{0,T,\eta}/4\widetilde{K}^2\right)\right\}
\end{aligned} \quad (24)$$

Remark 3.4 *The exponential integrability condition (19) on the coefficient σ_W in the SDE (15) is indispensable for the $W_1\mathbb{H}$ transportation inequalities in this theorem.*

We can explain parts 3. of Theorem 3.3 by following result which is an application to concentration empirical measure.

Corollary 3.5 *Let ψ be a (non empty) family of real Lipschitzian functions V on $\mathbb{R}^d$ with $||V||_{Lip}\leq\kappa$ and*

$$Y_T := \sup_{V\in\psi}\left(\frac{1}{T}\int_0^T V(X_s(x)) - \mu(V)\right).$$

Then, for all $T,u>0$ we have

$$\mathbb{P}(Y_T - \mathbb{E}Y_T > u) \leq \exp[-\phi_T^P(Tu)] \leq \exp[-TZ(u\widetilde{K})], \quad (25)$$

where $Z(u\widetilde{K}) := \sup_{\zeta\geq 0}\left\{u\widetilde{K}\zeta - \left(C(\zeta) + \zeta^2T^{2-2\eta}||\sigma_H(.,x(.))||^2_{0,T,\eta}/4\right)\right\}$.

Proof. First, we show that Y_T is measurable. We assume that $V(0)=0$ for all $V\in\psi$. Then we have by the ARZELA-ASCOLI theorem that $\{V_{\tilde{B}(0,R)}; V\in\psi\}$ is compact in $C_b(\tilde{B}(0,R))$, for any closed ball $\tilde{B}(0,R)$ centred at 0 of radius $R>0$ which, implies the measurability of Y_T on the event $\sup_{s\leq t}|X_s(x)|\leq R$. It remains to let $R\to+\infty$.

Now, we consider F and F_∞ defined on $C([0,T];\mathbb{R}^d)$ by

$$\begin{aligned}
F(\gamma) &:= \sup_{V\in\psi}\left|\frac{1}{T}\int_0^T(V(\gamma(t))dt - \mu(V))\right|\\
F_\infty(\gamma) &:= \sup_{t\in[0,T]}|\gamma(t)-\gamma(0)|.
\end{aligned}$$

The function F is $\kappa/\sqrt{T}$-Lipschitzian with respect to the L^1-metric. Hence

$$||F||_{Lip} = \sup_{\gamma_1\neq\gamma_2}\frac{d_{L^1}(F(\gamma_1),F(\gamma_2))}{d_{L^1}(\gamma_1,\gamma_2)} \leq \sup_{\gamma_1\neq\gamma_2}\sup_{V\in\psi}\frac{\frac{1}{T}\int_0^T|V(\gamma_1(t))-V(\gamma_2(t))|dt}{d_{L^1}(\gamma_1,\gamma_2)} \leq \frac{\kappa}{\sqrt{T}}.$$

Thus, we can apply theorem 3.3 (part 3) and lemma 3.1 to conclude that $Y_T = F(X(t,x))$ satisfies (25).

□

4 Malliavin calculus

The main purpose of this section is to introduce the Malliavin calculus relating to our case. Let $\Omega_1 = C_0([0,T];\mathbb{R}^d)$ be the Banach space of continuous functions, null at time 0, equipped with the supremum norm. Fix $H \in (\frac{1}{2},1)$. Let $\mathbb{P}$ be the unique probability measure on Ω_1 such that the canonical process $\{B_t^H; t \in [0,T]\}$ is an d-dimensional fractional Brownian motion with Hurst parameter H.

We denote by $\mathcal{E}$ the set of step functions on $[0,T]$ with values in $\mathbb{R}^d$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle(\mathbf{1}_{[0,t_1]},\ldots,\mathbf{1}_{[0,t_d]}),(\mathbf{1}_{[0,s_1]},\ldots,\mathbf{1}_{[0,s_d]})\rangle_{\mathcal{H}} = \sum_{i=1}^{d} R_H(t_i,s_i),$$

where

$$R_H(t,s) = \int_0^{t\wedge s} H_H(t,r)K_H(s,r)dr,$$

K_H is the square integrable kernel defined by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du \quad \forall t > s,$$

$c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}}$ and β denotes the Beta function.

The mapping $(\mathbf{1}_{[0,t_1]},\ldots,\mathbf{1}_{[0,t_d]}) \mapsto \sum_{i=1}^{d} B_{t_i}^i$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ spanned by B. We denote this isometry by $\varphi \mapsto B(\varphi)$.

Let $K_H^* : \mathcal{E} \to L^2([0,T];\mathbb{R}^d)$ be the operator defined by

$$K_H^*((\mathbf{1}_{[0,t_1]},\ldots,\mathbf{1}_{[0,t_d]})) = (K_H(t_1,.),\ldots,K_H(t_d,.)).$$

For any $\varphi_1,\varphi_2 \in \mathcal{E}$, $\langle\varphi_1,\varphi_2\rangle_{\mathcal{H}} = \langle K_H^*\varphi_1, K_H^*\varphi_2\rangle_{L^2([0,T];\mathbb{R}^d)} = \mathbb{E}(B(\varphi_1)B(\varphi_2))$ and then K_H^* provides an isometry between the Hilbert space $\mathcal{H}$ and a closed subspace of $L^2([0,T];\mathbb{R}^d)$.

Let $\mathcal{K}_H : L^2([0,T];\mathbb{R}^d) \to \mathcal{H}_H := \mathcal{K}_H\ (L^2([0,T];\mathbb{R}^d))$ be the operator defined by

$$(\mathcal{K}_H h)(t) := \int_0^t K_H(t,s)h(s)ds.$$

The space $\mathcal{H}_H$ is the fractional version of the Cameron-Martin space. In the case of a classical Brownian motion, $K_H(t,s) = \mathbf{1}_{[0,t]}(s)$, $\mathcal{K}_H^*$ is the identity map on $L^2([0,T];\mathbb{R}^d)$, and $\mathcal{H}_H$ is the space of absolutely continuous functions, vanishing at zero, with a square integrable derivative.

We denote by $\mathcal{R}_H = \mathcal{K}_H \circ \mathcal{K}_H^* : \mathcal{H} \to \mathcal{H}_H$ the operator

$$\mathcal{R}_H\varphi = \int_0^{\cdot} K_H(.,s)(\mathcal{K}_H^*\varphi)(s)ds.$$

We remark that for any $\varphi \in \mathcal{H}$, $\mathcal{R}_H\varphi$ is Hölder continuous of order H. Indeed,

$$(\mathcal{R}_H\varphi)^i(t) = \int_0^T (\mathcal{K}_H^*\mathbf{1}_{[0,t]})^i(s)(\mathcal{K}_H^*\varphi)^i(s)ds = \mathbb{E}[B_t^i B^i(\varphi)],$$

and consequently

$$\left|(\mathcal{R}_H\varphi)^i(t) - \mathcal{R}_H\varphi)^i(s)\right| \leq \left(\mathbb{E}(|B_t^i - B_s^i|^2)\right)^{1/2} ||\varphi||_{\mathcal{H}}|t-s|^H.$$

By Fubini's theorem and if $f \in C^\lambda(0,T)$ with $\lambda + H > 1$, $v \in L^2(0,T)$ then it holds that

$$\int_0^T f(r)d(\mathcal{K}_H v)_r = \int_0^T f(r)\left(\int_0^r c_H(r-t)^{H-\frac{3}{2}}r^{H-\frac{1}{2}}v(t)dt\right)dr. \qquad (26)$$

The integral in the left hand side of (26) is a Riemann-Stieltjes integral for Hölder functions. At last if $\varphi, \psi \in L^2([0,T];\mathbb{R}^d)$, then the scalar product on $\mathcal{H}$ has the integral form

$$\langle\varphi,\psi\rangle = H(2H-1)\int_0^T\int_0^T |s-t|^{2H-2}\langle\varphi(s),\psi(t)\rangle_{\mathrm{R}^d}dsdt,$$

and consequently for $\varphi \in L^2([0,T];\mathbb{R}^d)$it holds that

$$||\varphi||^2_{\mathcal{H}} \leq 2HT^{2H-1}||\varphi||_{L^2([0,T];\mathrm{R}^d)}. \qquad (27)$$

Let $C_b^\infty(\mathbb{R}^d)$ be the set of infinitely continuously differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth. If F is a smooth cylindrical random variable of the form $F = f(B(\varphi_1),\ldots,B(\varphi_d))$ for all $\varphi_i \in \mathcal{H}$, then we define the Malliavin derivative operator ∇F as the $\mathcal{H}$-valued random variable defined by

$$\begin{aligned}\langle\nabla F, h\rangle_{\mathcal{H}} &= \sum_{i=1}^d \partial_i f\left(B(\varphi_1),\ldots,B(\varphi_d)\right)\langle\varphi_i,h\rangle_{\mathcal{H}} \\ &= \frac{d}{d\epsilon}f\left(B(\varphi_1)+\epsilon\langle\varphi_1,h\rangle_{\mathcal{H}},\ldots,B(\varphi_d)+\epsilon\langle\varphi_d,h\rangle_{\mathcal{H}},\right)|_{\epsilon=0},\end{aligned} \qquad (28)$$

and one can easily see that $B(\varphi_1)(\omega + \epsilon\mathcal{R}_H h) = B(\varphi_1)(\omega) + \epsilon\langle\varphi_1, h\rangle_{\mathcal{H}}$.

We recall here that $\mathcal{D}^{k,p}$ is the closure of the space of smooth and cylindrical random variables with respect to the norm

$$||F||_{k,p} = \left[\mathbb{E}|F|^p + \sum_{j=1}^k \mathbb{E}||\nabla^j F||^p_{\mathcal{H}^\otimes}\right]^{\frac{1}{p}},$$

and $\mathcal{D}^{k,p}_{loc}$ is the set of random variables F such that there exist a sequence $\{(\Omega_n, F_n), n \geq 1\}$ such that $\Omega_n \uparrow \Omega_1$ a.s., $F_n \in \mathcal{D}^{k,p}$ and $F = F_n$ a.s. on Ω_n.

Proof of Theorem 3.2. Assume that $\mathbb{H}(Q|\mathbb{P}_x) < \infty$ for all probability measure Q on $C([0,T];\mathbb{R}^d)$ such that $Q \ll \mathbb{P}_x$.

Following [5], we first consider

$$\tilde{Q} = \frac{dQ}{d\mathbb{P}_x}(X)\mathbb{P}.$$

Here $\tilde{Q}$ is a probability measure on the space Ω_1 and

$$\begin{aligned}\mathbb{H}(\tilde{Q}|\mathbb{P}) &= \int_{\Omega_1} \ln\left(\frac{d\tilde{Q}}{d\mathbb{P}}\right) d\tilde{Q} = \int_{\Omega_1} \ln\left(\frac{dQ}{d\mathbb{P}_x}(X)\right) \frac{dQ}{d\mathbb{P}_x}(X) d\mathbb{P} \\ &= \int_{C([0,T];\mathbb{R}^d)} \ln\left(\frac{dQ}{d\mathbb{P}_x}(X)\right) \frac{dQ}{d\mathbb{P}_x}(X) d\mathbb{P}x = \mathbb{H}(Q|\mathbb{P}_x).\end{aligned}$$

Using [2], there exists a predictable process $N = (N_1(t), \dots, N_m(t))$, $t \in [0,T]$ such that

$$\mathbb{H}(Q|\mathbb{P}_x) = \mathbb{H}(\tilde{Q}|\mathbb{P}) = \frac{1}{2}\mathbb{E}_{\tilde{Q}} \int_0^T |N(t)|^2 dt.$$

By Girsanov's theorem, the process $(\tilde{B}_t)_{0\le t\le T}$ defined by

$$\tilde{B}_t = W_t - \int_0^t N(s) ds$$

is a Brownian motion under $\tilde{Q}$, and by the transfer principle, it is associated with the $\tilde{Q}$-fractional Brownian motion $(\tilde{B}^H_t)_{0\le t\le T}$ defined by

$$\begin{aligned}\tilde{B}^H_t &= \int_0^t K_H(t,s) d\tilde{B}_s = \int_0^t K_H(t,s) dW_s - (\mathcal{K}_H N)(t) \\ &= B^H_t - (\mathcal{K}_H N)(t).\end{aligned}$$

Thus, under $\tilde{Q}$, X satisfies the following SDE on $\mathbb{R}^d$

$$dX_t = x + b(t, X_t)dt + \sigma_W(t, X_t)dW_t + \sigma_H(t)d\tilde{B}^H_t + \sigma_H(t)d(\mathcal{K}_H N)(t), \quad x \in \mathbb{R}^d. \tag{29}$$

Now, let Y the $\tilde{Q}$-solution of of the following SDE on $\mathbb{R}^d$

$$dY_t = x + b(t, Y_t)dt + \sigma_W(t, Y_t)dW_t + \sigma_H(t)d\tilde{B}^H_t. \tag{30}$$

Then, the law of the process $(Y_t)_{0\le t\le T}$ is exactly $\mathbb{P}_x$ and (X, Y) is a coupling of $(Q, \mathbb{P}_x)$ under $\tilde{Q}$. Thus its follows that

$$\left[W_{1,d_{L^1}}(Q, \mathbb{P}_x)\right]^2 \le \mathbb{E}_{\tilde{Q}}\left(|d_{L^1}(X,Y)|^2\right) = \mathbb{E}_{\tilde{Q}}\left(\left[\int_0^T |X_t - Y_t| dt\right]^2\right).$$

Let us now estimate the distance between X and Y on $C([0,T];\mathbb{R}^m)$ and $C([0,T];\mathbb{R}^n)$ with respect to d_{L^1}. Using [12] the equations (29) and (30) can be considered as pathwise integral equations driven by α-Hölder functions with $\alpha < H$. We notice here that the Hölder regularity is straightforward for the driving function $\tilde{B}$ since it is a fractional Brownian motion under $\tilde{Q}$ (so has almost-surely α-Hölder trajectories for any $\alpha < H$). Moreover, since $\int_0^T |N(s)|^2 ds < \infty$ a.s., then $\mathcal{K}_H N \in C^H([0,T])$ a.s. (see [8], [12]).

We have

$$X_t - Y_s = \int_0^T \Big(b(t, X_t) - b(s, Y_s)\Big)dt + \int_0^T \Big(\sigma_W(t, X_t) - \sigma_W(s, Y_s)\Big)dW_t + \int_0^T \sigma_H(t)d(\mathcal{K}_H N)(t).$$

Using the change of variables formula for α-Hölder continuous function (see [17], Theorem 4.3.1) and the stability assumptions $(\mathbf{H}_1)$, $(\mathbf{H}_3)$ and $(\mathbf{H}_6)$ we obtain

$$
\begin{aligned}
|X_t - Y_s|^2 &= 2\sum_{i=1}^{d}\sum_{j=1}^{m}\int_0^T (X_t^i - Y_s^i)\sigma_H^{i,j}(t)d(\mathcal{K}_H N^j)(t) \\
&+ 2\int_0^T \langle X_t - Y_s, b(t,X_t) - b(s,Y_s)\rangle_{\mathbb{R}^d} dt \\
&+ 2\sum_{i=1}^{d}\sum_{k=1}^{n}\int_0^T (X_t^i - Y_s^i)\Big(\sigma_W^{i,k}(t,X_t) - \sigma_W^{i,k}(s,Y_s)\Big)dW_t^k \\
&\leq 2\sum_{i=1}^{d}\sum_{j=1}^{m}\int_0^T (X_t^i - Y_s^i)\sigma_H^{i,j}(t)d(\mathcal{K}_H N^j)(t) \\
&+ 2\sum_{i=1}^{d}\sum_{k=1}^{n}\int_0^T (X_t^i - Y_s^i)\Big(\sigma_W^{i,k}(t,X_t) - \sigma_W^{i,k}(s,Y_s)\Big)dW_t^k \\
&+ 2\bar{c}\int_0^T |t-s|^{\vartheta}dt + 2\bar{c}\int_0^T |X_t - Y_s|^2 dt.
\end{aligned} \tag{31}
$$

From (26) and the definition of the operator $\mathcal{K}_H^*$ as $(\mathcal{K}_H^*\varphi)(t) = \int_s^T \varphi(r)\frac{\partial K_H}{\partial r}(r,s)dr$, since $X - Y \in C^\alpha([0,T];\mathbb{R}^d)$ and N is a square-integrable process in $\mathbb{R}^m$, we obtain

$$
\begin{aligned}
&\int_0^T (X_t^i - Y_s^i)\sigma_H^{i,j}(t)d(\mathcal{K}_H N^j)(t) \\
&\quad = \int_0^T (X_t^i - Y_s^i)\sigma_H^{i,j}(t)\Big(\int_0^t c_H(t-\theta)^{H-\frac{3}{2}}t^{H-\frac{1}{2}}N^j(\theta)d\theta\Big)dt \\
&\quad = \int_0^T \Big(\int_\theta^T (X_t^i - Y_s^i)\sigma_H^{i,j}(t)c_H(t-\theta)^{H-\frac{3}{2}}t^{H-\frac{1}{2}}dt\Big)N^j(\theta)d\theta \\
&\quad = \int_0^T \mathcal{K}_H^*\Big((X^i - Y^i)\sigma_H^{i,j}\mathbf{1}_{[0,T]}\Big)(\theta)N^j(\theta)d\theta.
\end{aligned}
$$

On the other hand, if σ^* is the transpose matrix of σ, we obtain from the inequality (27)

$$
\begin{aligned}
&2\sum_{i=1}^{d}\sum_{j=1}^{m}\int_0^T (X_t^i - Y_s^i)\sigma_H^{i,j}(t)d(\mathcal{K}_H N^j)(t) \\
&\quad = 2\int_0^T \langle \mathcal{K}_H^*(\sigma_H^*(X-Y)\mathbf{1}_{[0,T]})(\theta), N(\theta)\rangle_{\mathbb{R}^m} d\theta \\
&\quad \leq 2||\mathcal{K}_H^*(\sigma_H^*(X-Y)\mathbf{1}_{[0,T]})||_{L^2(0,T)}||N||_{L^2(0,T)} \\
&\quad \leq 2||\sigma_H^*(X-Y)\mathbf{1}_{[0,T]}||_{\mathcal{H}}||N||_{L^2(0,T)} \\
&\quad \leq 2(2H)^{1/2}T^H||\sigma_H^*(X-Y)\mathbf{1}_{[0,T]}||_{L^2(0,T)}||N||_{L^2(0,T)} \\
&\quad \leq 2(2H)^{1/2}T^H||\sigma_H||_{\infty;[0,T]}||X-Y||_{L^2(0,T)}||N||_{L^2(0,T)}
\end{aligned}
$$

We report this estimation in (31) and using the fact that $2\epsilon ab \leq 4\epsilon^2a^2 + b^2$ with

$\epsilon = (HT^{2H}||\sigma_H||^2_{\infty;[0,T]}/(2\widetilde{K}))^{1/2}$ to get

$$\begin{aligned}
|X_t - Y_s|^2 &\leq 2(2H)^{1/2}T^H||\sigma_H||_{\infty;[0,T]}||X-Y||_{L^2(0,T)}||N||_{L^2(0,T)} \\
&+ 2\bar{c}\int_0^T |t-s|^{\vartheta}dt + 2\bar{c}\int_0^T |X_t - Y_s|^2 dt \\
&+ 2T||\sigma_W||_{\infty;[0,T]}||X-Y||_{L^2(0,T)} \\
&\leq HT^{2H}||\sigma_H||^2_{\infty;[0,T]}(2/\widetilde{K})\int_0^T |N(t)|^2 dt + 4\widetilde{K}\int_0^T |X_t - Y_s|^2 dt \\
&+ 2\widetilde{K}T^2 + 4T^2||\sigma_W||^2_{\infty;[0,T]},
\end{aligned}$$

when $\bar{c} = \widetilde{K}$.

By the Gronwall Lemma, it follows that for any $t > 0$

$$|X_t - Y_s|^2 \leq HT^{2H}||\sigma_H||^2_{\infty;[0,T]}(2/\widetilde{K})\int_0^T \exp[4\widetilde{K}(T-t)]|N(t)|^2 dt + 2\widetilde{K}T^2 + 4T^2||\sigma_W||^2_{\infty;[0,T]}.$$

So for the metric d_{L_1}, we may write that

$$\begin{aligned}
\Big[W_{1,d_{L_1}}(Q,\mathbb{P}_x)\Big]^2 &\leq \mathbb{E}_{\tilde{Q}}\int_0^T |X_t - Y_s|^2 dt \\
&\leq HT^{2H}||\sigma_H||^2_{\infty;[0,T]}(2/\widetilde{K}) \\
&\times \mathbb{E}_{\tilde{Q}}\int_0^T |N(t)|^2\Big(\int_t^T \exp[4\widetilde{K}(r-t)]dr\Big)dt + 2\widetilde{K}T^2 + 4T^2||\sigma_W||^2_{\infty;[0,T]}.
\end{aligned}$$

Since

$$\int_t^T \exp[4\widetilde{K}(r-t)]dr \leq \frac{1-e^{-\widetilde{K}T}}{\widetilde{K}},$$

we write $c_{\widetilde{K},T} = 1 - e^{\widetilde{K}T}$, with $\widetilde{K} > 0$, and consequently

$$\begin{aligned}
\Big[W_{1,d_{L_1}}(Q,\mathbb{P}_x)\Big]^2 &\leq \frac{4}{\widetilde{K}^2}HT^{2H}||\sigma_H||^2_{\infty;[0,T]}c_{\widetilde{K},T}\Big(\frac{1}{2}\mathbb{E}_{\tilde{Q}}\int_0^T |N(t)|^2 dt\Big) \\
&+ +4T^2||\sigma_W||^2_{\infty;[0,T]} \\
&\leq 2C_{T,H}\mathbb{H}(Q|\mathbb{P}_x),
\end{aligned}$$

with $C_{T,H} = \frac{2}{\widetilde{K}^2}HT^{2H}||\sigma_H||^2_{\infty;[0,T]}c_{\widetilde{K},T} + 4T^2||\sigma_W||^2_{\infty;[0,T]}$.

□

Proof of Theorem 3.3. To prove Theorem 3.3 we will require the following two lemmas.

Lemma 4.1 (See [10]) *Let the assumptions* $(\mathbf{H}_1)$-$(\mathbf{H}_2)$-$(\mathbf{H}_3)$, $(\mathbf{H}_4)$-$(\mathbf{H}_5)$ *and* $(\mathbf{H}_6)$ *be satisfied. For two different initial points* $x, y \in \mathbb{R}^d$, *the solutions* $X_t(x), X_t(y)$ *of the SDE (15) satisfy*

$$\mathbb{E}|X_t(x) - X_t(y)|^2 \leq \exp(-4\widetilde{K}\eta T^{2\eta}t)|x-y|^2, \quad t > 0. \tag{32}$$

If furthermore

$$||\sigma||_{Lip} := \sup_{x\neq y} \frac{\sqrt{||\sigma_W(x,.) - \sigma_W(y,.)||^2_{[0,T];\eta} + ||\sigma_H(x) - \sigma_H(y)||^2_{[0,T];\eta}}}{|x-y|} < \infty,$$

then there exist some positive constant $\widehat{K}$ *such that for* $L := 4\widetilde{K}\eta T^{2\eta} + ||\sigma||^2_{Lip}(4\widehat{K}^2 + 1)$

$$\mathbb{E} \sup_{0\leq\theta\leq\tau} |X_{t+\theta}(x) - X_{t+\theta}(y)|^2 \leq 2\exp(-4\widetilde{K}\eta T^{2\eta}t + 4L\tau)|x-y|^2, \quad t,\tau > 0. \quad (33)$$

Lemma 4.2 *Let the assumptions* $(\mathbf{H}_1)$-$(\mathbf{H}_2)$-$(\mathbf{H}_3)$*,* $(\mathbf{H}_4)$-$(\mathbf{H}_5)$*,* $(\mathbf{H}_6)$ *be satisfied and* $\nabla_s X_t$ *be defined as above. Then we have*

$$\mathbb{E}\Big[||\nabla_s X_t||^2_{[0,T];\eta}/\mathcal{F}_s\Big] \leq ||\sigma_H(X_s)||^2_{[0,T];\eta} \exp\Big[-4\widetilde{K}\eta T^{2\eta}|t-s|\Big], \quad \forall t,s \geq 0. \quad (34)$$

If moreover $||\sigma||_{Lip} < \infty$*, then with the same* L *as in Lemma (4.1), we have*

$$\mathbb{E}\Big[\sup_{\theta} ||\nabla_s X_\theta||^2_{[0,T];\eta}/\mathcal{F}_s\Big] \leq 2\exp\Big[2L|t-s|\Big]||\sigma_H(X_s)||^2_{[0,T];\eta}, \quad \theta > 0. \quad (35)$$

Proof. We explain here why the first inequality (34) holds. Let $\omega^j_H(t)$, $\omega^k_W(t)$ be respectively the j-th and the k-th coordinate of $\omega_H(t)$ and $\omega_W(t)$, which are a one-dimensional fractional Brownian motion and standard Brownian motion. We apply the chain rule for the Malliavin derivative operator ∇ (see [1]), we obtain for any $1 \leq k_1 \leq m$, $1 \leq \bar{k}_1 \leq m$ and $t,s \geq 0$,

$$\begin{aligned}\nabla_{s,k_1} X^i_t &= \sum_{l=1}^d \int_0^t \partial_l b_i(u,X_u)\nabla_{s,k_1}X^l_u du + \sigma^{ik_1}_H(s,X_s) + \sum_{j=1}^m\sum_{l=1}^d \int_0^t \partial_l \sigma^{ij}_H(u,X(u))\nabla_{s,k_1}X^l_u \omega^j_H(du) \\ &+ \sigma^{ik_1}_W(s,X_s) + \sum_{k=1}^n\sum_{l=1}^d \int_0^t \partial_l \sigma^{ik}_W(u,X(u))\nabla_{s,k_1}X^l_u \omega^k_W(du).\end{aligned}$$

Now we fix s and for $t > 0$, we set $Y := \nabla_s X_t$ which is an $m \times d$ matrix. We also denote by $Z_t := Y_t^* Y_t$ a $d \times d$ matrix. Then by Itô's formula we have

$$
\begin{aligned}
d||Y_t||^2_{\mathcal{H}} &= d||\nabla_s X_t||^2_{\mathcal{H}} \\
&= 2\sum_{k_1=1}^{m}\sum_{i=1}^{d}\nabla_{s,k_1}X_t^i d\nabla_{s,k_1}X_t^i + \sum_{k_2,k_3=1}^{m}\sum_{i_2,i_3=1}^{d} d\langle \nabla_{s,k_2}X^{i_2}, \nabla_{s,k_3}X^{i_3}\rangle_t \\
&= 2\sum_{k_1=1}^{m}\sum_{i,l=1}^{d}\nabla_{s,k_1}X_t^i\partial_l b_i(t,X_t)\nabla_{s,k_1}X_t^l dt + 2\sum_{k_1,j=1}^{m}\sum_{i,l=1}^{d}\nabla_{s,k_1}X_t^i\partial_l\sigma_H^{ij}(t,X_t)\nabla_{s,k_1}X_t^l\omega_H^j(dt) \\
&+ 2\sum_{\bar{k}_1,k=1}^{n}\sum_{i,l=1}^{d}\nabla_{s,\bar{k}_1}X_t^i\partial_l\sigma_W^{ik}(t,X_t)\nabla_{s,\bar{k}_1}X_t^l\omega_W^k(dt) \\
&+ \sum_{j=1}^{m}\Big(\sum_{k_1=1}^{m}\sum_{i,l=1}^{d}\nabla_{s,k_1}X_t^i\partial_l\sigma_H^{ij}(t,X_t)\nabla_{s,k_1}X_t^l\Big)^2 dt \\
&+ \sum_{k=1}^{n}\Big(\sum_{\bar{k}_1=1}^{n}\sum_{i,l=1}^{d}\nabla_{s,\bar{k}_1}X_t^i\partial_l\sigma_H^{ik}(t,X_t)\nabla_{s,\bar{k}_1}X_t^l\Big)^2 dt \\
&= 2\sum_{i,l=1}^{d} z_{il}(t)\partial_l b_i(t,X_t)dt + \sum_{j=1}^{m}\Big(\sum_{i,l=1}^{d} z_{il}(t)\partial_l\sigma_H^{ij}(t,X_t)\Big)^2 dt \\
&+ 2\sum_{j=1}^{m}\sum_{i,l=1}^{d} z_{il}(t)\partial_l\sigma_H^{ij}(t,X_t)\omega_H^j(dt) + \sum_{k=1}^{n}\Big(\sum_{i,l=1}^{d} z_{il}(t)\partial_l\sigma_W^{ik}(t,X_t)\Big)^2 dt \\
&+ 2\sum_{k=1}^{n}\sum_{i,l=1}^{d} z_{il}(t)\partial_l\sigma_W^{ik}(t,X_t)\omega_W^k(dt).
\end{aligned}
$$

Since Z_t is a non-negative definite $d\times d$ matrix, there exists a symmetric $d\times d$ matrix $\hat{Z}_t$ such that $Z_t = \hat{Z}_t^2$. Then,

$$
\begin{aligned}
d||\nabla_s X_t||^2_{\mathcal{H}} &= 2\sum_{i=1}^{d}\langle \hat{Z}_t^i, \nabla_s b\hat{Z}_t^i\rangle dt + \sum_{j=1}^{m}\sum_{i=1}^{d}\langle \hat{Z}_t^i, \nabla_s\sigma_H^j\hat{Z}_t^i\rangle^2 dt + \sum_{k=1}^{n}\sum_{i=1}^{d}\langle \hat{Z}_t^i, \nabla_s\sigma_W^k\hat{Z}_t^i\rangle^2 dt \\
&+ 2\sum_{j=1}^{m}\sum_{i,l=1}^{d} z_{il}(t)\partial_l\sigma_H^{ij}(t,X_t)\omega_H^j(dt) + 2\sum_{k=1}^{n}\sum_{i,l=1}^{d} z_{il}(t)\partial_l\sigma_W^{ik}(t,X_t)\omega_W^k(dt) \\
&\le -2\widetilde{K}\sum_{i=1}^{d}\langle \hat{Z}_t^i, \hat{Z}_t^i\rangle dt + M_t \\
&= -2\widetilde{K}||\hat{Z}_t||^2_{\mathcal{H}}dt + M_t \\
&= -2\widetilde{K}||Y_t||^2_{\mathcal{H}}dt + M_t \\
&\le -4\widetilde{K}\eta T^{2\eta}||\nabla_s X_t||^2_{[0,T];\eta}dt + M_t,
\end{aligned}
$$

where M is a local martingale. By applying the Gronwall's inequality and a localization procedure, we get

$$
\mathbb{E}\Big[||\nabla_s X_t||^2_{[0,T];\eta}/\mathcal{F}_s\Big] \le ||\sigma_H(X_s)||^2_{[0,T];\eta}\exp\Big[-4\widetilde{K}\eta T^{2\eta}|t-s|\Big].
$$

The proof of the inequality (35) is similar to that of (33) (for further details, we refer the reader to [10]).

□

Proof.(Theorem 3.3)
Part 1.

Let (X_0, Y_0) be a couple of $\mathbb{R}^d$-valued random variables of law ν_1, ν_2 respectively, independent of of B^H and W such that $\mathbb{E}|X_0 - Y_0| = W_{1,d}(\nu_1, \nu_d)$. Let X_t (resp. Y_t) be the solution of (15) with initial condition X_0 (resp. Y_0). (X_t, Y_t) constitutes a coupling of $\nu_1 P_t$ and $\nu_2 P_t$. Lemma (4.1) yields the estimate

$$\mathbb{E}\Big(|X_t - Y_t|/(X_0, Y_0)|\Big) \leq \Big[\mathbb{E}\Big(|X_t - Y_t|/(X_0, Y_0)\Big)\Big]^{1/2} \leq \exp(-2\widetilde{K}\eta T^{2\eta} t)|X_0 - Y_0|,$$

whence

$$W_{1,d}(\nu_1 P_t, \nu_2 P_t) \leq ||X_t - Y_t||_1 \leq \exp(-\widetilde{K}\eta T^{2\eta} t) W_{1,d}(\nu_1, \nu_2). \tag{36}$$

According to [16], for each $t > 0$, $\nu \to \nu P_t$ is a contraction mapping on the complete metric space $(E, W_{1,d})$, where $E = \{\nu; \mathbb{E}_\nu|x| < \infty\}$. Then, by the fixed point theorem for contraction mapping, P_T admits a unique invariant probability measure μ_t on $E_1 = \{\mu; \mathbb{E}_\nu|x|^2 < \infty\}$. So for each $s > 0$

$$W_1(\mu_t P_s, \mu_t) = W_1(\mu_t P_s P_t, \mu_t P_t) \leq \exp(-\widetilde{K}\eta T^{2\eta} t) W_1(\mu_t P_s, \mu_t)$$

which yields $\mu_t P_s = \mu_t$ and so $\mu_t = \mu_s$.
Now, for $t > 0$ fixed and any probability measure $\widetilde{\mu}$ of P_t, there exist some $x_0 \in \mathbb{R}^d$ such that $\frac{1}{n}\sum_{k_1=1}^{n} P_{nt}(x_0, dy) \to \widetilde{\mu}$ weakly. But $\frac{1}{n}\sum_{k_1=1}^{n} P_{nt}(x_0, dy) \to \mu$ in $W_{1,d}$-metric by (36), so $\widetilde{\mu} = \mu$. Thus μ_t is a unique invariant measure of P_t.
Finally, taking $\nu_1 = \nu, \nu_1 = \nu$ in (36) we obtain (20).

Part 2.

First, suppose that we are given a well defined measurable function F on Ω such that the difference operator $D_{t,u}F(\omega) := F(\omega + h_{t,u}) - F(\omega)$, $h_{t,x} \in C^\eta$ plays the role of the Malliavin calculus on the space Ω, the role of the Malliavin derivative operator on $\mathcal{H}$. Thus, with the above assumptions, the solution $X(t,\omega) = (X_t(x,\omega))_{t\geq 0}$ of the SDE (15) is a measurable mapping in to $C([0,T]; \mathbb{R}^d)$. Hence on $\mathbb{R}^d$, we admit that $X^{(t,u)}(x,\omega) := X(x, \omega + h_{t,u})$ satisfies

$$\begin{array}{rcl} X_s^{(t,u)}(x,\omega) &=& X_s(x,\omega), \quad \text{if } s < t; \\ X_s^{(t,u)}(x,\omega) &=& X_t(x,\omega) + \tilde{\sigma}_t(X_t(x,\omega)) + \int_t^s b(X_a^{(t,u)}(x,\omega), a)da + \int_t^s \sigma_H(X_a^{(t,u)}(x,\omega), a)\omega_H da \\ && + \int_t^s \sigma_W(X_a^{(t,u)}(x,\omega), a)\omega_W da, \quad \text{if } s > t, \end{array} \tag{37}$$

i.e. after time t, $(X_s^{(t,u)}(x,\omega))_{s\geq t}$ is the solution of the same SDE but with initial value $X_t^{(t,u)}(x,\omega) = X_s(x,\omega) + \tilde{\sigma}_t(X_t(x,\omega))$. Now for all real Lipschitzian function f on $\mathbb{R}^d$ with $||f||_{Lip} \leq 1$, we have

$$D_{t,u} f(X_T(x)) = 0, \quad \text{if } t > T,$$

and

$$|D_{t,u}f(X_T(x))| = |f(X_T^{(t,u)}(x)) - f(X_T(x))| \leq |X_T^{(t,u)}(x) - X_T(x)|, \quad \text{if } t < T.$$

Using Lemma 4.1, one easily deduces that

$$\begin{aligned}
\mathbb{E}\Big[|D_{t,u}f(X_T(x))|/\mathcal{F}_t\Big] &\leq \mathbb{E}\Big[|X_T^{(t,u)}(x) - X_T(x)|/\mathcal{F}_t\Big] \\
&\leq \exp\Big(-2\widetilde{K}\eta T^{2\eta}(T-t)\Big)|X_t^{(t,u)}(x) - X_t(x)| \\
&= \exp\Big(-2\widetilde{K}\eta T^{2\eta}(T-t)\Big)|\tilde{\sigma}_t(X_t)| \\
&\leq \exp\Big(-2\widetilde{K}\eta T^{2\eta}(T-t)\Big)||\sigma_W(t,X_t)||^2_{0,T,\eta}.
\end{aligned}$$

On the other hand,

$$|\nabla_s f(X_T(x))|^2 = \sum_{k=1}^{d}(\nabla_{s,k}f(X_T(x)))^2 \leq \sum_{k=1}^{d}\sum_{i=1}^{d}(\nabla_{s,k}X_T^i(x))^2 = ||\nabla_s X_T(x)||^2_{[0,T],\eta}.$$

Therefore by Lemma 4.2, we have for $s < T$

$$\mathbb{E}\Big[|\nabla_s f(X_T(x))|/\mathcal{F}_s\Big] \leq \mathbb{E}[||\nabla_s X_T||_{[0,T],\eta}/\mathcal{F}_s] \leq e^{-2\widetilde{K}\eta T^{2\eta}(T-s)}||\sigma_H(X_s(x))||_{[0,T],\eta},$$

hence, from lemma 3.2 ([16]) we have the concentration inequality for $t < T$

$$\begin{aligned}
&\mathbb{E}\exp[\zeta T^{2\eta}(f(X_T(x)) - P_T f(x))] \\
&\leq \exp\Big\{\int_0^T \mathbb{E}\Big[e^{\zeta T^{2\eta}e^{2\widetilde{K}\eta T^{2\eta}(T-t)}||\sigma_W||^2_{0,T,\eta}} - \zeta T^{2\eta}e^{-2\widetilde{K}\eta T^{2\eta}(T-t)}||\sigma_W||^2_{0,T,\eta} - 1\Big]dt \\
&+ \frac{||\sigma_H||^2_{0,T,\eta}T^{2-2\eta}\zeta^2}{4\widetilde{K}}(1 - e^{-2\widetilde{K}\eta T^{2\eta+1}})\Big\} \\
&= \exp\Big\{\int_0^T C(e^{-2\widetilde{K}T^{2\eta}t}T^{2\eta}\zeta)dt + \frac{||\sigma_H(.,x(.))||^2_{0,T,\eta}T^{2-2\eta}\zeta^2}{4\widetilde{K}}(1 - e^{-2\widetilde{K}\eta T^{2\eta+1}})\Big\}
\end{aligned}$$

Thus, the transportation inequality (21) follows by Lemma 3.1 and Fenchel's theorem under the condition on ϕ. Finally, the convexity of C (with $C(0) = 0$) implies $C(e^{-2\widetilde{K}T^{2\eta}t}T^{2\eta}\zeta) \leq C(T^{2\eta}\zeta)e^{-2\widetilde{K}T^{2\eta}t}$. Then

$$\begin{aligned}
\phi_T(u) &= \sup_{\zeta\geq 0}\left\{u\zeta T^{2\eta} - \int_0^T C(e^{-2\widetilde{K}T^{2\eta}t}T^{2\eta}\zeta)dt - \frac{||\sigma_H(.,x(.))||^2_{0,T,\eta}T^{2-2\eta}\zeta^2}{4\widetilde{K}}(1 - e^{-2\widetilde{K}\eta T^{2\eta+1}})\right\} \\
&\geq \sup_{\zeta\geq 0}\left\{u\zeta - C(T^{2\eta}\zeta)/\widetilde{K} - \frac{||\sigma_H(.,x(.))||^2_{0,T,\eta}T^{2-2\eta}\zeta^2}{4\widetilde{K}}\right\} \\
&= \frac{1}{\widetilde{K}}\sup_{\zeta\geq 0}\Big\{u\widetilde{K}\zeta T^{2\eta} - (C(\zeta T^{2\eta}) + \vartheta\zeta^2 T^{2-2\eta}||\sigma_H(.,x(.))||^2_{0,T,\eta}/2)\Big\}.
\end{aligned}$$

Now, letting T tend to ∞, we obtain (22) for the invariant measure μ (see [5]).

Part 3.

Let F be bounded d_{L^1}-Lipschitzian function on $C([0,T];\mathbb{R}^d)$ with $||F||_{Lip} \leq 1$. Then we have

$$|D_{t,u}F(X_{[0,T]}(x))| \leq d_{L^1}\Big(X^{t,u}_{[0,T]}(x), X_{[0,T]}(x)\Big) = \int_t^T |X^{t,u}_s(x) - X_s(x)|ds.$$

By (37) and Lemma (4.1) we get for $t < s$,

$$\mathbb{E}\Big(|D_{t,u}F(X_{[0,T]})|/\mathcal{F}_t\Big) \leq \int_t^T e^{-2\widetilde{K}\eta T^{2\eta}(s-t)}|\tilde{\sigma}_t(X_t)|dt \leq \frac{||\sigma_W(t,X_t)||^2_{0,T,\eta}}{\widetilde{k}}(1-e^{-2\widetilde{K}\eta T^{2\eta}(T-t)}).$$

On the other hand, $|\nabla_t F(X_{[0,T]}(x))| \leq \int_t^T ||\nabla_t X_r(x)||_{0,T,\eta}dr$, hence

$$\begin{aligned}\mathbb{E}\Big(|\nabla_t F(X_{[0,T]}(x))|/\mathcal{F}_t\Big) &\leq \int_t^T \mathbb{E}[||\nabla_t X_r(x)||_{0,T,\eta}/\mathcal{F}_r]dr \leq ||\sigma_H(X_r)||^2_{0,T,\eta}\int_t^T e^{-2\widetilde{K}\eta T^{2\eta}(T-r)}dr \\ &= \frac{||\sigma_H(X_r)||^2_{0,T,\eta}}{\widetilde{K}}(1-e^{-2\widetilde{K}\eta T^{2\eta}(T-r)}) \leq \frac{||\sigma_H(X_r)||^2_{0,T,\eta}}{\widetilde{K}}\end{aligned}$$

by (34).

Using Lemma 3.2 of [16] we obtain for all $\zeta \geq 0$

$$\mathbb{E}e^{\zeta(F(X_{[0,T]}(x)) - \mathrm{E}F(X_{[0,T]}(x)))} \leq$$
$$\exp\left(\int_0^T C((1-e^{-2\widetilde{K}T^{2\eta}t})T^{2\eta}\zeta/\widetilde{K})dt + ||\sigma_H||^2_{0,T,\eta}\int_0^T [(1-e^{-2\widetilde{K}\eta T^{2\eta}t})/\widetilde{K}]^2dt\right).$$

By Lemma 3.1 again, the function

$$\mathbb{R}_+ \ni u \to \sup\left\{u\zeta T^{2\eta} - \int_0^T C(e^{-2\widetilde{K}T^{2\eta}t}T^{2\eta}\zeta)dt - \frac{||\sigma_H||^2_{0,T,\eta}\zeta^2 T^{2-2\eta}}{4}\int_0^T\left[\frac{(1-e^{-2\widetilde{k}\eta T^{2\eta}t})}{\widetilde{K}}\right]^2dt\right\} = \phi^P_T(u$$

is a $W_1\mathbb{H}$-deviation function for $\mathbb{P}_{x,[0,T]}$ w.r.t the d_{L^1}-metric, which is exactly (23).

Finally, for the lower bound in (24) we use the fact that $(1-e^{-2\widetilde{K}T^{2\eta}t})/\widetilde{K} \leq 1/\widetilde{K}$ and $C(\zeta)$ is increasing in ζ, which implies that $C\Big[((1-e^{-2\widetilde{K}T^{2\eta}t})/\widetilde{K})\zeta T^{2\eta}\Big] \leq C\Big[(\zeta T^{2\eta})/\widetilde{K}\Big]$ and then $\int_0^T C[((1-e^{-2\widetilde{K}T^{2\eta}t})/\widetilde{K})\zeta T^{2\eta}] \leq TC[(\zeta T^{2\eta})/\widetilde{K}]$. Therefore

$$\phi^P_T(u) \geq \sup_{\zeta\geq 0}\left(u\zeta T^{2\eta} - T\Big(C(\zeta T^{2\eta}/\widetilde{K}) + \zeta^2 T^{2-2\eta}||\sigma_H(.,x(.))||^2_{0,T,\eta}/4\widetilde{K}^2\Big)\right).$$

□

References

[1] F. Biagini, Y. Hu, B. Øksendal, T. Zhang, Stochastic calculus for fractional Brownian motion and applications, Springer-Verlag London Limited 2008.

[2] S.G. Bobkov, F. Götze, Exponential integrability and transportation cost related to logarithmic Sobolev inequalities, J. Funct. Anal. 163 (1999) 1-28.

[3] B. Boufoussi, Y. Ouknine, On a SDE driven by a fractional Brownian motion and with monotone drift, Electron. Comm. Probab. 8 (2003) 122-134.

[4] L. Coutin, Z. Qian, Stochastic analysis, rough path analysis and fractional Brownian motions, Probab. Theory Related Fields 122 (2002) 108-140.

[5] H. Djellout, A. Guillin, L. Wu, Transportation cost-information inequalities for random dynamical systems and diffusions, Ann. Probab. 32 (3B) (2004) 2702-2732.

[6] N. Gozlan, C. Léonard, A large deviation approach to some transportation cost inequalities, Probab. Theory Related Fields 139 (2007) 235-283.

[7] T. Guendouzi, On the positive correlations in Wiener space via fractional calculus, Acta Universitatis Sapientiae, Mathematica, 2, 2 (2010) 206-214

[8] J. Guerra, D. Nualart, Stochastic differential equations driven by fractional Brownian motion and standard Brownian motion, Arxiv. (2008) 0801. 4963v1.

[9] Y. Hu, D. Nualart, X. Song, A singular stochastic differential equation driven by fractional Brownian motion, Stat. Probab. Lett 78 (2008) 2075-2085

[10] Y. Mishura, Stochastic calculus for fractional Brownian motion and related processes, Springer-Verlag Berlin Heidelberg 2008.

[11] I. Nourdin, T. Simon, On the absolute continuity of one-dimensional SDEs driven by a fractional Brownian motion, Statist. Probab. Lett. 76 (2006) 907-912.

[12] D. Nualart, B. Saussereau, Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion, Stoch. Proc. their App. 119 (2009) 391-409.

[13] D. Nualart, The Malliavin Calculus and Related Topics, 2nd ed., Springer, Berlin, 2005.

[14] F. Proske, The stochastic transport equation driven by Lévy white noise, Comm. Math. Sci. Vol 2, No. 4, (2004) 627-641

[15] M. Röckner, F.Y. Wang, Harnack and functional inequalities for generalized Mehler semigroups, J. Funct. Anal. 203 (2003), no. 1, 237-261.

[16] L. Wu, Transportation inequalities for stochastic differential equations of pure jumps. Preprint, Ann. Inst. H. Poincaré (2008).

[17] M. Zähle, Integration with respect to fractal functions and stochastic calculus I. Prob. Theory Relat. Fields 111, (1998) 333-374.