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Fuzzy EOQ model for deteriorating items with price dependent demand and time-varying holding cost

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Abstract

In this paper, a fuzzy economic order quantity (EOQ) model is formulated for single deteriorating item considering price dependent demand and time dependent holding cost. Shortages are allowed and partially backlogged with a variable rate dependent on the length of the waiting time for the next replenishment. For the Fuzzification of model, the demand, holding cost, unit purchase cost, deterioration rate, ordering cost, shortage cost and cost of lost sale are assumed to be trapezoidal fuzzy numbers. As a result, the profit function will be derived in fuzzy sense in order to obtain the optimal cycle length and the selling price. The graded mean integration method is used to defuzzify the profit function. Then, to illustrate the model as well as to test the validity of the model a numerical example is considered and solved. Finally, to study the effect of changes of different parameters on the average profit, order quantity, cycle length and selling price, sensitivity analysis are performed.

Keywords: Inventory, price dependent demand, partial backlogging, trapezoidal fuzzy number, graded mean representation method.

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1. Introduction

Demand is always one of the most influential factors in the decision making relating to inventory policy. In the existing literature of inventory control theory, it is observed that various forms of demand (like constant demand, price dependent demand etc.) have been studied by several researchers. In this connection, one may refer to the works of constant demand [Padmanabhan and Vrat, 1990], price dependent demand [Abad, 1996, 2001], time dependent demand [Sachan, 1984], and time and price dependent demand [Wee, 1995]. Jaggi and Aggarwal [1988] discussed a lot size model for exponentially deteriorating inventory with partial backlogged shortages.

Over the last few decades, several researchers have applied the fuzzy set theory and techniques to develop and solve the inventory problems. For example, [Park, 1987] and [Vujosevic, et al. 1996] extended the classical EOQ model by introducing the fuzziness of ordering cost and holding cost. [Chen and Wang, 1996] fuzzified the demand, ordering cost, inventory cost and backorder cost into trapezoidal fuzzy numbers in EOQ model with backorder. [Roy and Maiti,1997] presented a fuzzy EOQ model with demand-dependent unit cost under limited storage capacity considering different parameters as fuzzy sets with suitable membership function. [Chang, et al. 1998] presented a fuzzy model for inventory with backorder, where the backorder quantity was fuzzified as triangular fuzzy number. [Lee and Yao, 1998] and [Lin and Yao, 2000] discussed the production inventory problems, where [Lee and Yao, 1998] fuzzified the demand quantity and production and [Lin and Yao, 2000] fuzzified the production quantity per cycle, treating all the parameters as the triangular fuzzy numbers. [Yao, et al.2000] proposed the EOQ model in the fuzzy sense, considering the order quantity and also the demand as triangular fuzzy numbers. [Ouyang and Yao, 2002] presented a mixture inventory model with variable lead-time, where the annual demand was fuzzified as the triangular fuzzy number and as the statistic fuzzy number.

[Park, 1987] used the fuzzy set concept to treat the inventory problem with fuzzy inventory cost under arithmetic operations of extension principle. He examined the EOQ model from the fuzzy set theoretic perspective. In this model, trapezoidal fuzzy numbers were used for the ordering and inventory holding costs. [Chen, et al.1996] introduced the backorder fuzzy inventory model under function principle. [Chen, et al. 1996] discussed the fuzzy inventory model for crisp order quantity and also fuzzy order quantity with generalized trapezoidal fuzzy number. [Hsieh, 2002] introduced two fuzzy production inventory models with fuzzy parameters for crisp as well as fuzzy production quantity. By using the Graded Mean Integration Representation method of defuzzification, fuzzy production inventory cost is found and using the Extension of the Lagrangian method, they solved the problem with inequality constraint.

[Roy, 2008] discussed an inventory model considering the demand rate as a function of selling price and time dependent deterioration rate and holding cost. In his model, shortages were considered as fully backlogged. [Sahoo, et al. 2010] extended Roy's model by taking constant deterioration rate.

In this article, we have developed a infinite planning horizon inventory model for deteriorating items with price-dependent demand, time dependent holding cost, constant deterioration rate and stock-out cost (includes the backorder cost and lost sale cost) where the unsatisfied demand is partially backlogged with a variable rate dependent on the length of waiting time for the next replenishment. The demand, ordering cost, holding cost, unit cost, deterioration rate, shortage cost and lost sale cost are assumed to be trapezoidal fuzzy numbers to find the estimate of the average profit in fuzzy sense and then to derive the optimal cycle length and selling price. For this purpose, graded mean representation method is used to find the average fuzzy profit. Then the model has been illustrated with the help of a numerical example. Finally, sensitivity analysis have been performed the effect of changes of different parameters on the average profit, order quantity, cycle length and selling price.

2. Representation of Generalized Fuzzy Number

The approach of fuzzy set is an extension of classical set theory and it is used in fuzzy logic. In classical set theory, the membership of each element in relation to a set is assessed in binary terms according to a crisp condition; an element either belongs to or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of each element in relation to a set; this is discussed with the aid of a membership function. Fuzzy set is an extension of classical set theory since, for a certain universe, a membership function may act as an indicator function, mapping all elements to either 1 or 0, as in the classical notion.

[Chen and Hsieh, 1999] first proposed the concept of graded mean integration representation method which is based on the integral value of graded mean h-level of generalized fuzzy number for defuzzification of generalized fuzzy number to achieve the computational efficiency.

Trapezoidal Fuzzy Number

Trapezoidal Fuzzy Number is described as any fuzzy subset of the real line R, whose membership function $\mu_{\tilde{A}}$ satisfies the following conditions:

(1) μ_A is a continuous mapping from R to the closed interval [0, 1],

(2) $\mu_A = 0, -\infty < x \leq a$

(3) μ_A = L (x) is strictly increasing on [a, b],

(4) $\mu_A = w_A$, b ≤ x ≤ c,

(5) μ_A = R (x) is strictly decreasing on [c, d],

(6) $\mu_A = 0$, d ≤ x < ∞,

where $0 < w_A \leq 1$, and a, b, c, and d are real numbers.

Also this type of generalized fuzzy numbers be denoted as $\tilde{A}$ = (a, b, c, d; w_A) $_{LR}$,

when w_A =1, it can be simplified as $\tilde{A}$ = (a, b , c , d) $_{LR}$.

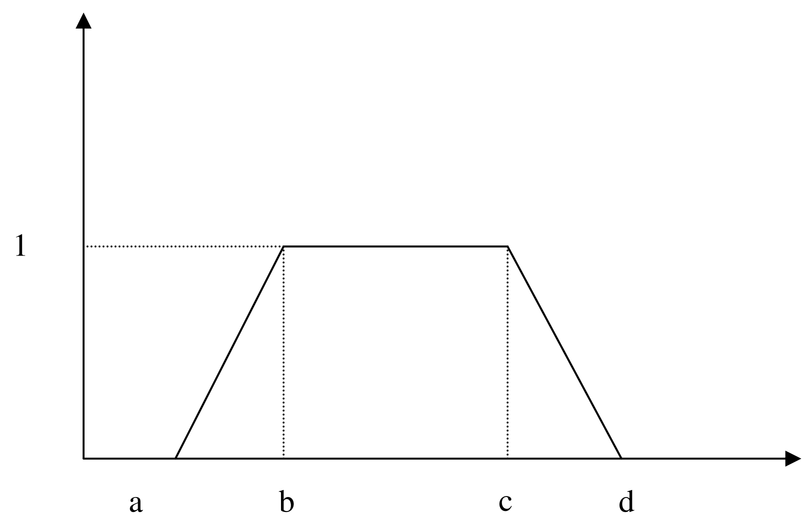

Figure- 1 Trapezoidal Fuzzy Number

Graded Mean Integration Representation

In graded mean integration representation method, L^{-1} and R^{-1} are the inverse functions of L and R respectively and the graded mean h-level value of generalized fuzzy number $\tilde{A}=$ (a, b, c, d; w_A) $_{LR}$ is $h\ (L^{-1}\ (h)+\ R^{-1}\ (h))/2$. Then the graded mean integration representation of $\tilde{A}$ is P ($\tilde{A}$) with grade w_A where

$$P(\tilde{A})=\frac{\int_0^{w_A} h\left(\frac{L^{-1}(h)+R^{-1}(h)}{2}\right)dh}{\int_0^{w_A} hdh}$$ with $0 < h \le w_A$ and $0 < w_A \le 1$.

If $\tilde{A}=(a,b,c,d)$ is a trapezoidal fuzzy number then the graded mean integration representation of $\tilde{A}$ by above formula is

$$p(A)=1/2\frac{\int_0^1 h[a+d+h(b-a-d+c)]dh}{\int_0^1 hdh}=\frac{a+2b+2c+d}{6}.$$

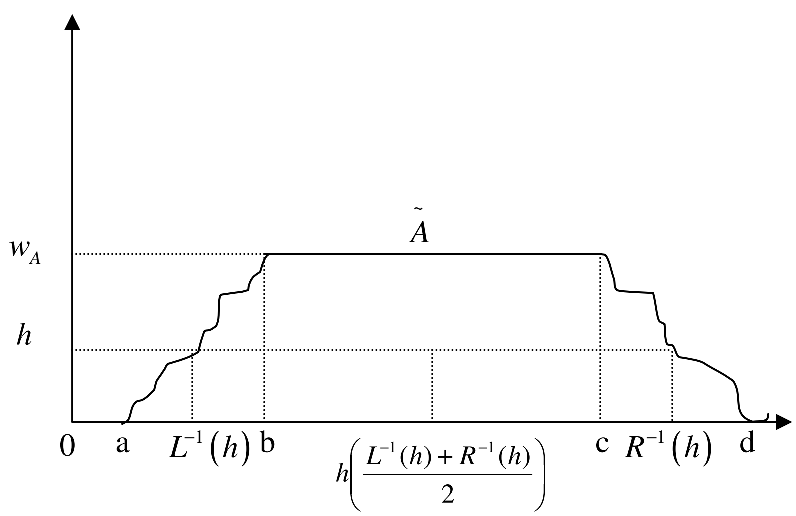

Figure -2 The graded mean h-level value of generalized fuzzy number $\tilde{A}=(a,b,c,d;w_A)$.

3. Notations and Assumptions

The mathematical model has been developed on the basis of the following notations and assumptions.

(i) The demand rate is a known function of selling price only i.e., $D(p) = a - bp, a, b > 0.$

(ii) Replenishment is instantaneous, but its size is finite.

(iii) The time horizon of the inventory system is infinite.

(iv) Lead time is constant.

(v) Shortages are allowed and partially backlogged.

(vi) During stock-out period, the backlogging rate is variable and is dependent on the length of the waiting time for next replenishment. So that the backlogging rate for the negative inventory is $\frac{1}{1+\delta(T-t)}$; where $\delta(>0)$ denotes the backlogging parameter and $(T - t)$ is waiting time during $t_1 \le t \le T$.

(vii) A is the ordering cost per order.

(viii) C is the unit purchase cost .

(ix) $h(t)\ (= h + \alpha t)$ is the holding cost per unit per unit time where $0 < \alpha < 1, h > 0.$

(x) θ is the constant deterioration rate, $0 < \theta << 1$.

(xi) S is the shortage cost per unit per unit time.

(xii) L is the lost sale cost per unit per unit time.

(xiii) p is the selling price per unit item.

(xiv) T is the cycle length.

(xv) t_1 is the time point when the inventory level of stock-in period reaches to zero.

(xvi) $I(t)$ is the inventory level at time t.

(xvii) q is the maximum inventory level at $t = 0$.

(xviii) IB is the maximum backordered units during stock-out period.

(xix) Q is the economic order quantity for the inventory cycle.

(xx) $P(T, p)$ is the average profit.

(xxi) $\tilde{A}$ is the fuzzy ordering cost per order.

(xxii) $\tilde{C}$ is the fuzzy unit purchase cost.

(xxiii) $\tilde{h}(t)\left(=\tilde{h}+\tilde{\alpha}t\right)$ is the fuzzy holding cost per unit per unit time.

(xxiv) $\tilde{\theta}$ is the fuzzy deterioration rate.

(xxv) $\tilde{S}$ is the fuzzy shortage cost per unit per unit time.

(xxvi) $\tilde{L}$ is the fuzzy lost sale cost per unit per unit time.

(xxvii) $\tilde{D}(p)\left(=\tilde{a}-\tilde{b}\,p\right)$ is the fuzzy demand rate.

(xxviii) $\tilde{P}(T,p)$ is the fuzzy average profit.

(xxix) $P_{dG}(T,p)$ is the defuzzify value of $\tilde{P}(T,p)$.

4. Model Formulation

A typical behavior of the inventory in a cycle is depicted in the following Figure-3.

4.1 Crisp Model

During the period $(0,t_1)$, the inventory depletes due to the cumulative effects of demand and deterioration. Hence, the inventory level at any instant of time during $(0,t_1)$ is described by the differential equation as follows:

$$\frac{dI(t)}{dt}+\theta I(t)=-(a-bp) \qquad 0\le t\le t_1 \tag{1}$$

with the condition $I(t_1)=0$. The solution of differential equation (1) is given by

$$I(t)=(a-bp)\left\{(t_1-t)+\theta\left(\frac{t_1^2}{2}+\frac{t^2}{2}-tt_1\right)+\theta^2\left(\frac{t_1^3}{6}-\frac{t^3}{6}-\frac{t_1^2t}{2}+\frac{t_1t^2}{2}\right)\right\} \tag{2}$$

(By neglecting the higher powers of θ like θ^3 and so on).

At time $t = t_1$, the inventory level reaches to zero and thereafter t_1 shortages occur. During the interval (t_1, T), the inventory level depends on the demand only. In that case only a fraction of the demand is backlogged. Inventory level during (t_1, T) can be represented by the differential equation as follows:

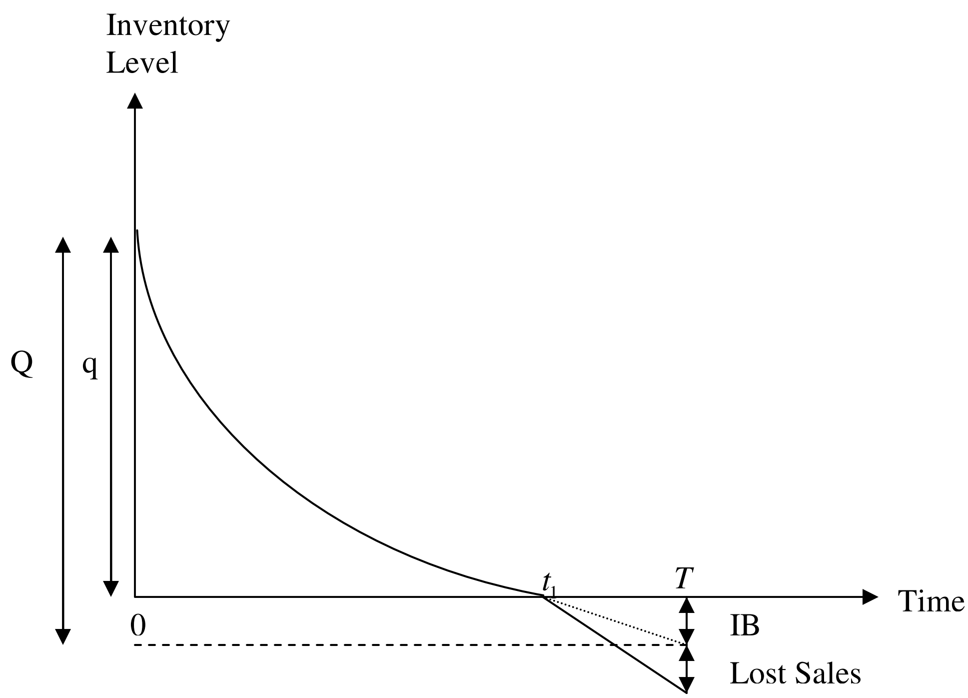

Figure-3 Pictorial Representation of Inventory system

$$\frac{dI(t)}{dt} = -\frac{(a-bp)}{1+\delta(T-t)} \qquad t_1 \le t \le T \tag{3}$$

with $I(t_1) = 0$.

The solution of the differential equation (3) is given by

$$I(t) = \frac{(a-bp)}{\delta}\left\{\log\left\{1+\delta(T-t)\right\} - \log\left\{1+\delta(T-t_1)\right\}\right\} \tag{4}$$

The maximum level of positive inventory is given by

$$q = I(0) = (a-bp)\left\{t_1 + \frac{\theta t_1^2}{2} + \frac{\theta^2 t_1^3}{6}\right\} \tag{5}$$

The maximum number of backordered units is given by

$$IB = -I(T) = \frac{(a-bp)}{\delta}\log\{1+\delta(T-t_1)\} \tag{6}$$

Hence, the economic order quantity is given by

$$Q = q + IB = (a-bp)\left\{t_1 + \frac{\theta t_1^2}{2} + \frac{\theta^2 t_1^3}{6}\right\} + \frac{(a-bp)}{\delta}\log\{1+\delta(T-t_1)\} \tag{7}$$

The total cost per cycle consists of following cost components.

(i) Ordering cost per cycle

$$OC = A \tag{8}$$

(ii) Inventory holding cost over the cycle is given by

$$HC = \int_0^{t_1}(h+\alpha t)\, I(t)dt$$

$$= (a-bp)h\left\{\frac{t_1^2}{2} + \frac{\theta t_1^3}{6} + \frac{\theta^2 t_1^4}{24}\right\} + (a-bp)\alpha\left\{\frac{t_1^3}{6} + \frac{\theta t_1^4}{24} + \frac{\theta^2 t_1^5}{120}\right\} \tag{9}$$

(iii) Shortage cost over the cycle is given by

$$SC = -S\int_{t_1}^{T}\frac{(a-bp)}{\delta}\left\{\log\{1+\delta(T-t)\} - \log\{1+\delta(T-t_1)\}\right\}dt$$

$$= \frac{(a-bp)S}{\delta^2}\left\{\delta(T-t_1) - \log\{1+\delta(T-t_1)\}\right\} \tag{10}$$

(iv) Cost due to lost sale over the cycle is given by

$$LS = L\int_{t_1}^{T}(a-bp)\left\{1 - \frac{1}{1+\delta(T-t)}\right\}dt$$

$$= \frac{(a-bp)L}{\delta}\left\{\delta(T-t_1) - \log\{1+\delta(T-t_1)\}\right\} \tag{11}$$

Hence the average profit is given by

$$P(T,t_1,p) = \left[p(a-bp)t_1 + \frac{(a-bp)p}{\delta}\log\{1+\delta(T-t_1)\} - (A + CQ + HC + SC + LS)\right]/T$$

$$P(T,t_1,p)=\frac{1}{T}\left[p(a-bp)t_1+\frac{(a-bp)p}{\delta}\log\{1+\delta(T-t_1)\}-\left\{\begin{array}{l}A+C(a-bp)\left\{\left(t_1+\frac{\theta t_1^2}{2}+\frac{\theta^2 t_1^3}{6}\right)+\frac{\log\{1+\delta(T-t_1)\}}{\delta}\right\}\\+(a-bp)h\left\{\frac{t_1^2}{2}+\frac{\theta t_1^3}{6}+\frac{\theta^2 t_1^4}{24}\right\}+(a-bp)\alpha\left\{\frac{t_1^3}{6}+\frac{\theta t_1^4}{24}+\frac{\theta^2 t_1^5}{120}\right\}\\+\frac{(a-bp)(S+\delta L)}{\delta^2}\{\delta(T-t_1)-\log\{1+\delta(T-t_1)\}\}\end{array}\right\}\right]$$

Let us assume that the stock-in period of the proposed inventory system is a proper fraction of cycle length i.e., $t_1 = vT$ where $0 < v < 1$. Now using $t_1 = vT$ in the above expression of $P(T,t_1,p)$, we have

$$P(T,p)=\frac{1}{T}\left[p(a-bp)vT+\frac{(a-bp)p}{\delta}\log\{1+\delta(T-vT)\}-\left\{\begin{array}{l}A+C(a-bp)\left\{\left(vT+\frac{\theta v^2T^2}{2}+\frac{\theta^2 v^3T^3}{6}\right)+\frac{\log\{1+\delta(T-vT)\}}{\delta}\right\}+\\(a-bp)\left\{h\left(\frac{v^2T^2}{2}+\frac{\theta v^3T^3}{6}+\frac{\theta^2 v^4T^4}{24}\right)+\alpha\left(\frac{v^3T^3}{6}+\frac{\theta v^4T^4}{24}+\frac{\theta^2 v^5T^5}{120}\right)\right\}\\+\frac{(a-bp)(S+\delta L)}{\delta^2}\{\delta(T-vT)-\log\{1+\delta(T-vT)\}\}\end{array}\right\}\right] \quad (12)$$

4.2 Fuzzy Model

We take $\tilde{a}=(a_1,a_2,a_3,a_4)$, $\tilde{b}=(b_1,b_2,b_3,b_4)$, $\tilde{A}=(A_1,A_2,A_3,A_4)$, $\tilde{\theta}=(\theta_1,\theta_2,\theta_3,\theta_4)$,

$\tilde{h}=(h_1,h_2,h_3,h_4)$, $\tilde{\alpha}=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $\tilde{C}=(C_1,C_2,C_3,C_4)$, $\tilde{S}=(S_1,S_2,S_3,S_4)$,

$\tilde{L}=(L_1,L_2,L_3,L_4)$ are as trapezoidal fuzzy numbers.

Hence the average profit in fuzzy sense is given by

$$\tilde{P}(T,p)=\frac{1}{T}\left[p\left(\tilde{a}-\tilde{b}p\right)vT+\frac{\left(\tilde{a}-\tilde{b}p\right)p}{\delta}\log\{1+\delta(T-vT)\}-\left\{\begin{array}{l}\tilde{A}+\tilde{C}\left(\tilde{a}-\tilde{b}p\right)\left\{\left(vT+\frac{\tilde{\theta}v^2T^2}{2}+\frac{\tilde{\theta}^2v^3T^3}{6}\right)+\frac{\log\{1+\delta(T-vT)\}}{\delta}\right\}+\left(\tilde{a}-\tilde{b}p\right)\\ \tilde{h}\left\{\frac{v^2T^2}{2}+\frac{\tilde{\theta}v^3T^3}{6}+\frac{\tilde{\theta}^2v^4T^4}{24}\right\}+\left(\tilde{a}-\tilde{b}p\right)\tilde{\alpha}\left\{\frac{v^3T^3}{6}+\frac{\tilde{\theta}v^4T^4}{24}+\frac{\tilde{\theta}^2v^5T^5}{120}\right\}\\ +\frac{\left(\tilde{a}-\tilde{b}p\right)\left(\tilde{S}+\delta\tilde{L}\right)}{\delta^2}\left\{\delta(T-vT)-\log\{1+\delta(T-vT)\}\right\}\end{array}\right\}\right] \tag{13}$$

Now we use Graded Mean defuzzification Method to convert the fuzzy average profit function in the corresponding crisp one.

By Graded mean defuzzification Method, we get the profit function as follows:

$$P_{dG}(T,p)=\frac{1}{6}\left[X_1+2X_2+2X_3+X_4\right] \tag{14}$$

where

$$X_1=p\left(a_1-b_4p\right)\left(v+\frac{\log\{1+\delta(T-vT)\}}{\delta T}\right)-\frac{1}{T}\left(\begin{array}{l}A_4+C_4\left(a_4-b_1p\right)\left\{\left(vT+\frac{\theta_4v^2T^2}{2}+\frac{\theta_4^2v^3T^3}{6}\right)+\frac{\log\{1+\delta(T-vT)\}}{T}\right\}+\\ (a_4-b_1p)\left\{h_4\left(\frac{v^2T^2}{2}+\frac{\theta_4v^3T^3}{6}+\frac{\theta_4^2v^4T^4}{24}\right)+\alpha_4\left(\frac{v^3T^3}{6}+\frac{\theta_4v^4T^4}{24}+\frac{\theta_4^2v^5T^5}{120}\right)\right\}\\ +\frac{(a_4-b_1p)(S_4+\delta L_4)}{\delta^2}\left\{\delta(T-vT)-\log\{1+\delta(T-vT)\}\right\}\end{array}\right)$$

$$X_2 = p\left(a_2 - b_3 p\right)\left(v + \frac{\log\left\{1+\delta\left(T - vT\right)\right\}}{\delta T}\right) - \frac{1}{T}\left(\begin{array}{l} A_3 + C_3\left(a_3 - b_2 p\right)\left\{\left(vT + \frac{\theta_3 v^2 T^2}{2} + \frac{\theta_3^{\,2} v^3 T^3}{6}\right) + \frac{\log\left\{1+\delta\left(T - vT\right)\right\}}{T}\right\} + \\ (a_3 - b_2 p)\left\{h_3\left(\frac{v^2T^2}{2} + \frac{\theta_3 v^3 T^3}{6} + \frac{\theta_3^{\,2} v^4 T^4}{24}\right) + \alpha_3\left(\frac{v^3T^3}{6} + \frac{\theta_3 v^4 T^4}{24} + \frac{\theta_3^{\,2} v^5 T^5}{120}\right)\right\} \\ + \frac{(a_3 - b_2 p)(S_3 + \delta L_3)}{\delta^2}\left\{\delta\left(T - vT\right) - \log\left\{1+\delta\left(T - vT\right)\right\}\right\} \end{array}\right)$$

$$X_3 = p\left(a_3 - b_2 p\right)\left(v + \frac{\log\left\{1+\delta\left(T - vT\right)\right\}}{\delta T}\right) - \frac{1}{T}\left(\begin{array}{l} A_2 + C_2\left(a_2 - b_3 p\right)\left\{\left(vT + \frac{\theta_2 v^2 T^2}{2} + \frac{\theta_2^{\,2} v^3 T^3}{6}\right) + \frac{\log\left\{1+\delta\left(T - vT\right)\right\}}{T}\right\} + \\ (a_2 - b_3 p)\left\{h_4\left(\frac{v^2T^2}{2} + \frac{\theta_2 v^3 T^3}{6} + \frac{\theta_2^{\,2} v^4 T^4}{24}\right) + \alpha_4\left(\frac{v^3T^3}{6} + \frac{\theta_2 v^4 T^4}{24} + \frac{\theta_2^{\,2} v^5 T^5}{120}\right)\right\} \\ + \frac{(a_2 - b_3 p)(S_2 + \delta L_2)}{\delta^2}\left\{\delta\left(T - vT\right) - \log\left\{1+\delta\left(T - vT\right)\right\}\right\} \end{array}\right)$$

$$X_4 = p\left(a_4 - b_1 p\right)\left(v + \frac{\log\left\{1+\delta\left(T - vT\right)\right\}}{\delta T}\right) - \frac{1}{T}\left(\begin{array}{l} A_1 + C_1\left(a_1 - b_4 p\right)\left\{\left(vT + \frac{\theta_1 v^2 T^2}{2} + \frac{\theta_1^{\,2} v^3 T^3}{6}\right) + \frac{\log\left\{1+\delta\left(T - vT\right)\right\}}{T}\right\} + \\ (a_1 - b_4 p)\left\{h_1\left(\frac{v^2T^2}{2} + \frac{\theta_1 v^3 T^3}{6} + \frac{\theta_1^{\,2} v^4 T^4}{24}\right) + \alpha_1\left(\frac{v^3T^3}{6} + \frac{\theta_1 v^4 T^4}{24} + \frac{\theta_1^{\,2} v^5 T^5}{120}\right)\right\} \\ + \frac{(a_1 - b_4 p)(S_1 + \delta L_1)}{\delta^2}\left\{\delta\left(T - vT\right) - \log\left\{1+\delta\left(T - vT\right)\right\}\right\} \end{array}\right)$$

Here, our objective is to maximize the average profit $P_{dG}(T, p)$. The necessary conditions for maximizing the profit are given by

$$\frac{\partial P_{dG}(T, p)}{\partial T} = 0 \text{ and } \frac{\partial P_{dG}(T, p)}{\partial p} = 0 \tag{15}$$

Equations (15) are equivalent to the following equations

$$\frac{1}{6}\frac{p\left(a_1 - b_4 p\right)(1-v)}{1+\delta\left(T - vT\right)T} - \frac{1}{6}\frac{p\left(a_1 - b_4 p\right)\log\left\{1+\delta\left(T - vT\right)\right\}}{\delta T^2} + \frac{1}{6T^2}$$

$$\left(\begin{array}{l} A_4 + C_4\left(a_4 - b_1 p\right)\left\{vT + \frac{\theta_4 v^2 T^2}{2} + \frac{\theta_4^{\,2} v^3 T^3}{6} + \frac{\log\left\{1+\delta\left(T - vT\right)\right\}}{\delta}\right\} + (a_4 - b_1 p)h_4\left\{\frac{v^2T^2}{2} + \frac{\theta_4 v^3 T^3}{6} + \frac{\theta_4^{\,2} v^4 T^4}{24}\right\} \\ +(a_4 - b_1 p)\alpha_4\left\{\frac{v^3T^3}{6} + \frac{\theta_4 v^4 T^4}{24} + \frac{\theta_4^{\,2} v^5 T^5}{120}\right\} + \frac{(a_4 - b_1 p)(S_4 + \delta L_4)}{\delta^2}\left\{\delta\left(T - vT\right) - \log\left\{1+\delta\left(T - vT\right)\right\}\right\} \end{array}\right)$$

$$-\frac{1}{6T}\left(\begin{array}{l}C_4(a_4-b_1p)\left\{v+\theta_4v^2T+\frac{\theta_4^2v^3T^2}{2}+\frac{(1-v)}{1+\delta(T-vT)T}\right\}+(a_4-b_1p)h_4\left\{v^2T+\frac{\theta_4v^3T^2}{2}+\frac{\theta_4^2v^4T^3}{6}\right\}\\+(a_4-b_1p)\alpha_4\left\{\frac{v^3T^2}{2}+\frac{\theta_4v^4T^3}{6}+\frac{\theta_4^2v^5T^4}{24}\right\}+\frac{(a_4-b_1p)(S_4+\delta L_4)}{\delta^2}\left\{\delta(1-v)-\frac{\delta(1-v)}{1+\delta(T-vT)}\right\}\end{array}\right)$$

$$+\frac{1}{3}\frac{p(a_2-b_3p)(1-v)}{1+\delta(T-vT)T}-\frac{1}{3}\frac{p(a_2-b_3p)\log\{1+\delta(T-vT)\}}{\delta T^2}+\frac{1}{3T^2}$$

$$\left(\begin{array}{l}A_3+C_3(a_3-b_2p)\left\{vT+\frac{\theta_3v^2T^2}{2}+\frac{\theta_3^2v^3T^3}{6}+\frac{\log\{1+\delta(T-vT)\}}{\delta}\right\}+(a_3-b_2p)h_3\left\{\frac{v^2T^2}{2}+\frac{\theta_3v^3T^3}{6}+\frac{\theta_3^2v^4T^4}{24}\right\}\\+(a_3-b_2p)\alpha_3\left\{\frac{v^3T^3}{6}+\frac{\theta_3v^4T^4}{24}+\frac{\theta_3^2v^5T^5}{120}\right\}+\frac{(a_3-b_2p)(S_3+\delta L_3)}{\delta^2}\{\delta(T-vT)-\log\{1+\delta(T-vT)\}\}\end{array}\right)$$

$$-\frac{1}{3T}\left(\begin{array}{l}C_3(a_3-b_2p)\left\{v+\theta_3v^2T+\frac{\theta_3^2v^3T^2}{2}+\frac{(1-v)}{1+\delta(T-vT)T}\right\}+(a_3-b_2p)h_3\left\{v^2T+\frac{\theta_3v^3T^2}{2}+\frac{\theta_3^2v^4T^3}{6}\right\}\\+(a_3-b_2p)\alpha_3\left\{\frac{v^3T^2}{2}+\frac{\theta_3v^4T^3}{6}+\frac{\theta_3^2v^5T^4}{24}\right\}+\frac{(a_3-b_2p)(S_3+\delta L_3)}{\delta^2}\left\{\delta(1-v)-\frac{\delta(1-v)}{1+\delta(T-vT)}\right\}\end{array}\right)$$

$$+\frac{1}{3}\frac{p(a_3-b_2p)(1-v)}{1+\delta(T-vT)T}-\frac{1}{3}\frac{p(a_3-b_2p)\log\{1+\delta(T-vT)\}}{\delta T^2}+\frac{1}{3T^2}$$

$$\left(\begin{array}{l}A_2+C_2(a_2-b_3p)\left\{vT+\frac{\theta_2v^2T^2}{2}+\frac{\theta_2^2v^3T^3}{6}+\frac{\log\{1+\delta(T-vT)\}}{\delta}\right\}+(a_2-b_3p)h_2\left\{\frac{v^2T^2}{2}+\frac{\theta_2v^3T^3}{6}+\frac{\theta_2^2v^4T^4}{24}\right\}\\+(a_2-b_3p)\alpha_2\left\{\frac{v^3T^3}{6}+\frac{\theta_2v^4T^4}{24}+\frac{\theta_2^2v^5T^5}{120}\right\}+\frac{(a_2-b_3p)(S_2+\delta L_2)}{\delta^2}\{\delta(T-vT)-\log\{1+\delta(T-vT)\}\}\end{array}\right)$$

$$-\frac{1}{3T}\left(\begin{array}{l}C_2(a_2-b_3p)\left\{v+\theta_2v^2T+\frac{\theta_2^2v^3T^2}{2}+\frac{(1-v)}{1+\delta(T-vT)T}\right\}+(a_2-b_3p)h_2\left\{v^2T+\frac{\theta_2v^3T^2}{2}+\frac{\theta_2^2v^4T^3}{6}\right\}\\+(a_2-b_3p)\alpha_2\left\{\frac{v^3T^2}{2}+\frac{\theta_2v^4T^3}{6}+\frac{\theta_2^2v^5T^4}{24}\right\}+\frac{(a_2-b_3p)(S_2+\delta L_2)}{\delta^2}\left\{\delta(1-v)-\frac{\delta(1-v)}{1+\delta(T-vT)}\right\}\end{array}\right)$$

$$+\frac{1}{6}\frac{p(a_4-b_1p)(1-v)}{1+\delta(T-vT)T}-\frac{1}{6}\frac{p(a_4-b_1p)\log\{1+\delta(T-vT)\}}{\delta T^2}+\frac{1}{6T^2}$$

$$\left(\begin{array}{l}A_1+C_1(a_1-b_4p)\left\{vT+\frac{\theta_1v^2T^2}{2}+\frac{\theta_1^2v^3T^3}{6}+\frac{\log\{1+\delta(T-vT)\}}{\delta}\right\}+(a_1-b_4p)h_1\left\{\frac{v^2T^2}{2}+\frac{\theta_1v^3T^3}{6}+\frac{\theta_1^2v^4T^4}{24}\right\}\\+(a_1-b_4p)\alpha_1\left\{\frac{v^3T^3}{6}+\frac{\theta_1v^4T^4}{24}+\frac{\theta_1^2v^5T^5}{120}\right\}+\frac{(a_1-b_4p)(S_1+\delta L_1)}{\delta^2}\{\delta(T-vT)-\log\{1+\delta(T-vT)\}\}\end{array}\right)$$

$$-\frac{1}{6T}\left(\begin{array}{l}C_1(a_1-b_4p)\left\{v+\theta_1v^2T+\frac{\theta_1^2v^3T^2}{2}+\frac{(1-v)}{1+\delta(T-vT)T}\right\}+(a_1-b_4p)h_1\left\{v^2T+\frac{\theta_1v^3T^2}{2}+\frac{\theta_1^2v^4T^3}{6}\right\}\\+(a_1-b_4p)\alpha_1\left\{\frac{v^3T^2}{2}+\frac{\theta_1v^4T^3}{6}+\frac{\theta_1^2v^5T^4}{24}\right\}+\frac{(a_1-b_4p)(S_1+\delta L_1)}{\delta^2}\left\{\delta(1-v)-\frac{\delta(1-v)}{1+\delta(T-vT)}\right\}\end{array}\right)=0$$

(16)

$$
\left[\begin{array}{l}
\frac{1}{6}\left(a_1-b_4p\right)v-\frac{1}{6}pb_4v+\frac{1}{6}\frac{\left(a_1-b_4p\right)\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}-\frac{1}{6}\frac{pb_4\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}-\\
\frac{1}{6T}\left\{\begin{array}{l}-C_4b_1\left(\frac{1}{2}\theta_4v^2T^2+\frac{1}{6}\theta_4{}^2v^3T^3+vT+\frac{\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}\right)-h_4b_1\left(\frac{1}{2}v^2T^2+\frac{1}{6}\theta_4v^3T^3+\frac{1}{24}\theta_4{}^2v^4T^4\right)-\\
\alpha_4b_1\left(\frac{1}{6}v^3T^3+\frac{1}{24}\theta_4v^4T^4+\frac{1}{120}\theta_4{}^2v^5T^5\right)-\frac{\left(S_4+\delta L_4\right)b_1\left(\delta\left(T-vT\right)-\log\left(1+\delta\left(T-vT\right)\right)\right)}{\delta^2}\end{array}\right\}\\
+\frac{1}{3}\left(a_2-b_3p\right)v-\frac{1}{3}pb_3v+\frac{1}{3}\frac{\left(a_2-b_3p\right)\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}-\frac{1}{3}\frac{pb_3\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}-\\
\frac{1}{3T}\left\{\begin{array}{l}-C_3b_2\left(\frac{1}{2}\theta_3v^2T^2+\frac{1}{6}\theta_3{}^2v^3T^3+vT+\frac{\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}\right)-h_3b_2\left(\frac{1}{2}v^2T^2+\frac{1}{6}\theta_3v^3T^3+\frac{1}{24}\theta_3{}^2v^4T^4\right)-\\
\alpha_3b_2\left(\frac{1}{6}v^3T^3+\frac{1}{24}\theta_3v^4T^4+\frac{1}{120}\theta_3{}^2v^5T^5\right)-\frac{\left(S_3+\delta L_3\right)b_2\left(\delta\left(T-vT\right)-\log\left(1+\delta\left(T-vT\right)\right)\right)}{\delta^2}\end{array}\right\}\\
+\frac{1}{3}\left(a_3-b_2p\right)v-\frac{1}{3}pb_2v+\frac{1}{3}\frac{\left(a_3-b_2p\right)\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}-\frac{1}{3}\frac{pb_2\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}-\\
\frac{1}{3T}\left\{\begin{array}{l}-C_2b_3\left(\frac{1}{2}\theta_2v^2T^2+\frac{1}{6}\theta_2{}^2v^3T^3+vT+\frac{\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}\right)-h_2b_3\left(\frac{1}{2}v^2T^2+\frac{1}{6}\theta_2v^3T^3+\frac{1}{24}\theta_2{}^2v^4T^4\right)-\\
\alpha_2b_3\left(\frac{1}{6}v^3T^3+\frac{1}{24}\theta_2v^4T^4+\frac{1}{120}\theta_2{}^2v^5T^5\right)-\frac{\left(S_2+\delta L_2\right)b_3\left(\delta\left(T-vT\right)-\log\left(1+\delta\left(T-vT\right)\right)\right)}{\delta^2}\end{array}\right\}\\
\frac{1}{6}\left(a_4-b_1p\right)v-\frac{1}{6}pb_1v+\frac{1}{6}\frac{\left(a_4-b_1p\right)\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}-\frac{1}{6}\frac{pb_1\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}-\\
\frac{1}{6T}\left\{\begin{array}{l}-C_1b_4\left(\frac{1}{2}\theta_1v^2T^2+\frac{1}{6}\theta_1{}^2v^3T^3+vT+\frac{\log\left\{1+\delta\left(T-vT\right)\right\}}{\delta T}\right)-h_1b_4\left(\frac{1}{2}v^2T^2+\frac{1}{6}\theta_1v^3T^3+\frac{1}{24}\theta_1{}^2v^4T^4\right)-\\
\alpha_1b_4\left(\frac{1}{6}v^3T^3+\frac{1}{24}\theta_1v^4T^4+\frac{1}{120}\theta_1{}^2v^5T^5\right)-\frac{\left(S_1+\delta L_1\right)b_4\left(\delta\left(T-vT\right)-\log\left(1+\delta\left(T-vT\right)\right)\right)}{\delta^2}\end{array}\right\}
\end{array}\right]=0 \tag{17}
$$

Equations (16) and (17) will give the optimal values of T and p. From the values of T and p the optimal value of the average profit $P_{dG}(T,p)$ can be determined by using the expression (14) for the profit function provided they satisfy the sufficient conditions for maximization of $P_{dG}(T,p)$ as follows:

$$\frac{\partial^2 P_{dG}(T,p)}{\partial T^2} < 0, \frac{\partial^2 P_{dG}(T,p)}{\partial p^2} < 0 \tag{18}$$

and

$$\frac{\partial^2 P_{dG}(T,p)}{\partial T^2}\frac{\partial^2 P_{dG}(T,p)}{\partial p^2} - \left(\frac{\partial^2 P_{dG}(T,p)}{\partial T \partial p}\right)^2 > 0 \tag{19}$$

The second derivatives of the net profit $P_{dG}(T,p)$ are highly non-linear and it is a formidable task to prove the concavity theoretically. Thus, the concavity of the net profit has been shown graphically (cf. Figure- 4) for a numerical example.

5. Numerical Example

For Crisp Model Let us consider an inventory system with the following data:
$A = 100, C = 50, h = 10, a = 100, \theta = .08, S = 12, \alpha = .1, v = .95, \delta = .5, L = 15$.

The solution of the crisp model with given values is as follows:

Total Profit $P(T,p)$ = 2502.38, Selling price (p)=127.08, Time (T) = 0.6438, Time (t_1) =0.6116, Q=24.

For Fuzzy Model, when $\tilde{a}, \tilde{b}, \tilde{A}, \tilde{C}, \tilde{\alpha}, \tilde{S}, \tilde{\theta}, \tilde{h}, \tilde{L}$ all are trapezoidal fuzzy numbers

Let us take $\tilde{a} = (96,98,102,104), \tilde{b} = (.46,.48,.52,.54), \tilde{C} = (46,48,52,54)$,

$\tilde{A} = (96,98,102,104)$, $\tilde{\theta} = (.04,.06,.10,.12), \tilde{h} = (6,8,12,14), \tilde{\alpha} = (.06,.08,.12,.14)$,

$\tilde{S} = (8,10,14,16)$, $\tilde{L} = (11,13,17,19)$ as trapezoidal fuzzy numbers.

The solution of fuzzy model can be determined by graded mean representation method.
Hence, the solution of fuzzy model is

Total Profit $P_{dG}(T,p)$ = 2474.59, Selling price (p)=126.91, Time (T) = .6230, Time (t_1) = .5918, Q=23.

The three dimensional total profit per unit time graph is shown in Figure 4 by plotting p in the range of [75,175] and T in the range of [.3, 1] .The graph given in figure 4 indicates that total profit per unit time is strictly concave.

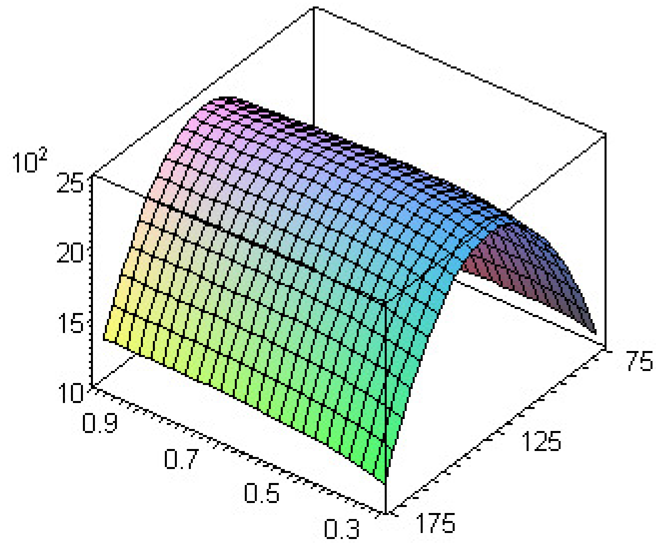

Figure-4 Total profit per unit time

6. Sensitivity Analysis

To investigate the effect of changes on different parameters like, a (the location parameter of demand), A (the ordering cost), C (the purchase cost), h (location parameter of holding cost), θ (the deterioration rate) and v on the cycle length, selling price, ordering quantity and average profit, sensitivity analysis has been performed numerically. This analysis has been carried out by changing -20% to +20% for one parameter except v keeping the other parameters as same. The results have been shown in Table-1. However, in Table-2, the sensitivity analysis of v has been performed considering the values from 0.75 to 0.95.

Table-1: Sensitivity Analysis of parameters *a, A, C, h,* θ

Parameters	%Changes	% changes in T	p	Q	$P_{dG}(T, p)$
A	-20	16.60	-15.50	-13.04	-50.65
	-10	7.34	-7.76	-4.35	-27.38
	10	-6.01	7.79	8.70	31.48
	20	-11.03	15.59	13.04	67.04
A	-20	-10.50	-0.17	-8.69	1.37
	-10	-5.10	-0.08	-4.34	0.67
	10	4.85	0.08	4.34	-0.63
	20	9.47	0.15	13.04	-1.24
C	-20	-0.27	-4.03	8.69	15.65
	-10	-0.18	-2.02	4.34	7.69
	10	0.27	2.02	0	-7.43
	20	0.63	4.03	-4.34	-14.59
H	-20	7.09	-0.13	8.69	0.87
	-10	3.37	-0.06	4.34	0.43
	10	-3.05	0.06	0	-0.42
	20	-5.83	0.43	-4.34	-0.82
θ	-20	3.02	-0.05	4.34	0.37
	-10	1.48	-0.02	4.34	0.18
	10	-1.41	0.02	0	-0.18
	20	-2.78	0.05	0	-0.36

From Table 1, the following inferences can be made:

(i) The cycle length T is insensitive with respect to the parameter C whereas it is slightly sensitive with respect to θ and moderately sensitive for the changes of parameters a, A and h.

(ii) The selling price p is insensitive with respect to the parameters A, h and θ, slightly sensitive with the changes of C and moderately sensitive with the changes of a.

(iii) The order quantity Q is moderately sensitive for the changes of parameters a, A, C, h and less sensitive with respect to θ.

(iv) The average profit $P_{dG}(T,p)$ is highly sensitive with respect to a. However, the rate of increase of the average profit is more than the decrease of the same. On the other hand, average profit is insensitive with the changes of h and θ, less sensitive with respect to the parameter A and moderately sensitive with the changes of parameter C.

Table-2: Sensitivity Analysis of parameter v

v	T	p	Q	$P_{dG}(T,p)$
0.75	0.6758	126.46	25	2496.91
0.80	0.6739	126.57	25	2497.35
0.85	0.6638	126.68	25	2493.71
0.90	0.6464	126.80	24	2486.06

From Table-2, it is observed that the average profit is higher for *v*=0.80 than that for other values of *v*. In numerical example, *v* is considered as 0.95and the corresponding optimal profit is 2474.59. With respect to that result, for all the values of *v*, the average profit is higher. For lower value of *v*, the stock-in period will be reduced and shortage period will be increased. As a result, the value of cycle length, order quantity as well as average profit will be increased.

7. Conclusion

In the present study, a crisp inventory model is developed with constant deterioration, price dependent demand and time varying holding cost and partial backlogged shortages. Thereafter, to develop the corresponding fuzzy model, trapezoidal fuzzy numbers have been used to represent the uncertainty in all the parameters namely, demand, ordering cost, holding cost, purchase cost, deterioration rate, shortage cost and lost sale cost. Now for Defuzzification, the well known Graded Mean Integration method has been employed to find the average profit and also to derive the optimal order quantity, the profit function has been maximized. To show the validity of the model a numerical example has been considered and solved. From the numerical example, it is observed that the optimal profit of fuzzy model is lesser than that of crisp one. The reason behind this is due to uncertainty of several parameters. Hence we conclude that the average profit will be reduced when uncertainties are accounted in large manner. Finally, a comprehensive sensitivity analysis is also conducted to show the effects of changes of the key parameters on the optimal order quantity, selling price, cycle length and average profit.

The proposed model can be extended in several ways. For example, one may consider the effect of inflation and time value of money by taking discount rate.

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References

1. Abad, P.L., (1996) Optimal pricing and lot- sizing under condition of perishability and partial backlogging. Management Science, vol. 42, pp.1093-1104.
2. Abad, P.L., (2001) Optimal price and order size for a reseller under partial backlogging. Computer and Operations Research, vol. 28, pp.53-65.
3. Chang, S.C., Yao, J.S. and Lee, H.M., (1998) Economic reorder point for fuzzy backorder quantity. European Journal of Operational Research, vol. 109, pp.183–202.
4. Chen, S. H. and Hsieh, C. H., (1999) Graded mean integration representation of generalized fuzzy numbers. Journal of Chinese Fuzzy Systems Association, vol. 5, nr.2, pp. 1-7.
5. Chen, S.H. and Wang, C.C., (1996) Backorder fuzzy inventory model under functional principle. Information Sciences, vol. 95, pp.71–9.
6. Hsieh, C.H., (2002) Optimization of fuzzy production inventory models. Information Sciences, vol. 146, nr. 1-4, pp. 29-40.IEEE.
7. Jaggi, C.K. and Aggarwal, S.P., (1988) Optimum inventory level with partial backlogging. Journal of Information and Optimization Sciences, vol. 9, nr. 3, pp. 437-444.
8. Lee, H.M. and Yao, J.S., (1998) Economic production quantity for fuzzy demand quantity and fuzzy production quantity. European Journal of Operational Research, vol.109, pp. 203–211.

9. Lin, D.C. and Yao, J.S., (2000) Fuzzy economic production for production inventory. Fuzzy Sets and Systems, vol. 111, pp. 465–95.
10. Ouyang, L.Y. and Yao. J.S., (2002) A minimax distribution free procedure for mixed inventory model involving variable lead-time with fuzzy demand. Computers and Operations Research, vol. 29, pp. 471–87.
11. Padmanabhan, G. and Vrat, P., (1990) Inventory model with a mixture of backorders and lost sales. International Journal of System Sciences, vol. 21, pp. 1721-1726.
12. Park, K., (1987) Fuzzy-set theoretic interpretation of economic order quantity. IEEE Transactions on Systems, Man, and Cybernetics SMC-17, pp. 1082-1084.
13. Roy, A., (2008) An inventory model for deteriorating items with price dependent demand and time varying holding cost. Advanced Modeling and Optimization, vol. 10, nr. 1.
14. Roy, T.K. and Maiti, M., (1997) A fuzzy EOQ model with demand-dependent unit cost under limited storage capacity. European Journal of Operational Research, vol. 99, pp. 425–432.
15. Sachan, R.S., (1984) On policy inventory model for deteriorating item with time proportional demand. Journal of the Operational Research Society, vol. 35, pp. 1013-1019.
16. Sahoo, N.K., Sahoo, C.K. and Sahoo, S.K., (2010) An inventory model for constant deteriorating items with price dependent demand and time varying holding cost. International Journal of Computer Sciences and Communication, vol. 1, pp. 267-271.
17. Vujosevic, M., Petrovic, D. and Petrovic, R., (1996) EOQ Formula when Inventory Cost is Fuzzy. International Journal Production Economics, vol.45, pp. 499-504.
18. Wee, H.M., (1995) Joint pricing and replenishment policy for deteriorating inventory with declining market. International Journal Production Economics, vol. 40, pp.163-171.
19. Yao, J.S., Chang, S.C. and Su, J.S., (2000) Fuzzy inventory without backorder for fuzzy order quantity and fuzzy total demand quantity. Computers and Operations Research, vol. 27, pp. 935–962.
20. Zadeh, L., (1965) Fuzzy sets, Information and Control. Vol.8, pp.338-353.