AMO - Advanced Modeling and Optimization, Volume 13, Number 3, 2011

A UNIFORMLY CONVERGENT METHOD BY NON STANDARD FINITE DIFFERENCE METHOD ON ARBITRARY MESHES FOR A SYSTEM OF SINGULARLY PERTURBED SEMILINEAR CONVECTION-DIFFUSION

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ABSTRACT. In this paper, a numerical solution for a system of singularly perturbed semilinear convection-diffusion is studied. The scheme is based on locally exact schemes or on locally Green's functions. It is proved that the numerical scheme has first order accuracy, which is uniform with respect to the perturbation parameters. A monotone iterative method is applied to computing the nonlinear difference scheme. Numerical results confirms the theory of the method.

1. INTRODUCTION

Consider the following system of l coupled singularly perturbed convection-diffusion equations: Find $\mathbf{u} = (u_1, \ldots, u_l) \in (c^2(0,1) \cap c[0,1])$ such that

$$L\mathbf{u} := -E\mathbf{u}'' + B\mathbf{u}' + C(x, \mathbf{u}) = \mathbf{f}(x), \tag{1.1}$$

$x \in \Omega = (0,1)$, $\mathbf{u}(0) = \mathbf{u}(1) = 0$, with $E = diag\{\varepsilon_1, \ldots, \varepsilon_l\}$, where $0 < \varepsilon_i << 1$ for $i = 1, \ldots, l$ are known small positive diffusion coefficients, and $\mathbf{f} = [f_i(x)]_{i=1}^{l}$ is a vector-valued right hand side and $C(x, \mathbf{u}) = (c_1(x, u_1(x), \ldots, u_l(x)), \ldots, c_l(x, u_1(x), \ldots, u_l(x))^T$, $u_i(x) \in c(0,1)$ and c_i are nonlinear functions for $i = 1, \ldots, l$. Suppose that the functions c_i, b_i and f_i for $i = 1, \ldots, l$, are sufficiently smooth. Furthermore, we shall assume that B is diagonal with diagonal elements $b_i(x)$ for $i = 1, \ldots, l$, and define

$$\beta_k = \min_{x \in [0,1]} |b_k(x)| > 0 \quad for \quad k = 1, \ldots, l. \tag{1.2}$$

We assume that $b_i(x) > 0$ for $i = 1, \ldots, l$. In other words, we assume that the ith equation of problem (1.1) has a strong boundary layer at $x = 1$, [15]. Suppose that the matrix

2000 *Mathematics Subject Classification.* 65L10, 65L20, 65L70, 65H10, 15A12.

Key words and phrases. Convection-diffusion, Non standard finite difference, Boundary layer, Uniform convergence, Monotone iterative method.

AMO - Advanced Modeling and Optimization. ISSN: 1841-4311.

$C_{\mathbf{u}} = [\frac{\partial c_i}{\partial u_j}]_{i,j=1}^{l}$ satisfies the condition

$$C_* \leq C_{\mathbf{u}} \leq C^*, \tag{1.3}$$

where $C_* = [c_{*ij}]_{i,j=1}^{l}$ and $C^* = [c_{ij}^*]_{i,j=1}^{l}$ are M-matrices and c_{*ij}, c_{ij}^* are constants. Recall that for two matrixs A and B, we write $A \leq B$ if $a_{ij} \leq b_{ij}$ for all i and j. Note that if in problem (1.1), $C(x, \mathbf{u}) = A\mathbf{u}$, then we have a linear version of singularly perturbed convection-diffusion. Linss and Dresden [11], Linss [12], Linss and Madden [13], Madden and Stynes [14] and Gracia and Lisbona [9] have done some works for linear version of proplem (1.1). Bellew and O'Riordan [3], Cen [6], Amiraliyev [1] and Andreev [2] used the finite difference method for a coupled system of two singularly perturbed convection-diffusion equations. In one dimension with discontinous data has been investigated in [4]. And in linear version we have some results in [5] and [8]. Our goal is to construct an ε-uniform numerical method for solving problem on arbitrary meshes by applying non standard finite difference, that is, a numerical method which generates ε-uniformly convergent numerical approximations to the solution. The paper is organized as follows: In Section 2, we establish some a priori estimates of the solution and its first derivatives. In Section 3, we construct a numerical method by applying the non standard finite difference method. In Section 4 we prove uniform convergence of the numerical method on arbitrary nonuniform meshes. In Section 5, we construct a monotone iterative method for solving the nonlinear difference scheme and prove that the iterative converges ε-uniformly to the solution of problem (1.1). In the last section numerical results are presented, which are in agreement with the theoretical results.

2. Properties of the continuous problem

To estimate the error in our difference approximation, we shall require some bounds for the derivatives of the solution of problem (1.1), we assume that

$$\sum_{j=1}^{l} \frac{\partial c_i}{\partial u_j} \geq 0, \ \frac{\partial c_i}{\partial u_i} > 0 \ \ and \ \ \frac{\partial c_i}{\partial u_j} \leq 0 \ \ for \ \ i \neq j, \ x \in [0,1] \ \ and \ \ i, j = 1, \ldots, l, \tag{2.1}$$

and strict inequality hold at least for one k, i.e.,

$$\sum_{j=1}^{l} \frac{\partial c_k}{\partial u_j} > 0. \tag{2.2}$$

By the mean-value theorem we have

$$C(x,\mathbf{u}) = C(x,0) + C_{\mathbf{u}}\mathbf{u}(x). \tag{2.3}$$

By substituting (2.3) in (1.1), we have

$$-E\mathbf{u}'' + B\mathbf{u}' + C_{\mathbf{u}}\mathbf{u}(x) = \mathbf{f}(x) - C(x,\mathbf{0}). \tag{2.4}$$

Lemma 2.1. *If* $\mathbf{u} = (u_1(x), \dots, u_l(x))^T$, $L\mathbf{u} \geq 0 (\leq 0)$ *in* Ω *and* $\mathbf{u}(0)$, $\mathbf{u}(1) \geq 0(\leq 0)$ *then* $\mathbf{u}(x) \geq 0 (\leq 0)$ *in* Ω .

Proof. Let $u_i(x)$ be minimum at t_i for $i = 1, \dots, l$, i.e, $u_i(t_i) = \min\limits_{x\in\Omega} u_i(x)$ and also assuming

$$u_j(t_j) = \min_{i=1,\dots,l} u_i(t_i),$$

If $u_j(t_j) \geq 0$ the lemma is proved. So let $u_j(t_j) < 0$. If $u_j(t_j) = u_k(t_j)$ for $k = 1, \dots, l$, then it follows that $\mathbf{u}'(t_j) = 0$ and $\mathbf{u}''(t_j) \geq 0$. By (2.4)

$$L\mathbf{u}|_{t_j} := -E\mathbf{u}''(t_j) + B\mathbf{u}'(t_j) + C_{\mathbf{u}}\mathbf{u}(t_j) \leq C_{\mathbf{u}}\mathbf{u}(t_j) = u_k(t_j) C_{\mathbf{u}}.\mathbf{1}.$$

In this case according to (2.2) since $u_k(t_j) < 0$, the kth component of $C_{\mathbf{u}}\mathbf{u}(t_j)$ is negative, which is a contradiction to the assumption of the lemma. If there is a k with $1 \leq k \leq l$ such that $u_j(t_j) < u_k(t_j)$ then

$$\begin{aligned} & -\varepsilon_j u_j''(t_j) + b_j(t_j)u_j'(t_j) + \sum_{m=1}^{l} \frac{\partial c_j}{\partial u_m}(t_j) u_m(t_j) \\ = \; & -\varepsilon_j u_j''(t_j) + \sum_{m=1}^{l} \frac{\partial c_j}{\partial u_m}(t_j) u_j(t_j) + \sum_{m=1, m\neq j}^{l} \frac{\partial c_j}{\partial u_m}(t_j)(u_m(t_j) - u_j(t_j)) \\ & \leq \max_{m=1,\dots,l}(u_m(t_j) - u_j(t_j)) \sum_{m=1,m\neq j}^{l} \frac{\partial c_j}{\partial u_m}(t_j) + u_j(t_j) \sum_{m=1}^{l} \frac{\partial c_j}{\partial u_m}(t_j). \end{aligned}$$

If $\sum_{m=1,m\neq j}^{l} \frac{\partial c_j}{\partial u_m}(t_j) < 0$, it is obvious that the right hand side of the above inequality is negative. If $\sum_{m=1,m\neq j}^{l} \frac{\partial c_j}{\partial u_m}(t_j) = 0$, then since $\frac{\partial c_j}{\partial u_j} > 0$, we have $\sum_{m=1}^{l} \frac{\partial c_j}{\partial u_m}(t_j) > 0$. Thus the right hand side is negative and again we reach a contradiction. So the lemma is proved. ☐

I. Boglaev [4] has proved the following lemma.

Lemma 2.2. *Let* $\mathbf{u}$ *be the solution of (1.1) and suppose(1.2), (2.1) and (2.2) hold. Then for* $x \in [0,1]$ *and* $n = 0,1$

$$|u_k^{(n)}(x)| \leq \begin{cases} C[1+\varepsilon_k^{-n}\exp(-\frac{\beta_k x}{\varepsilon_k})] & b_k \leq -\beta_k, \\ C[1+\varepsilon_k^{-n}\exp(-\frac{\beta_k(1-x)}{\varepsilon_k})] & b_k \geq \beta_k, \end{cases}$$

here and throughout the paper, C *denotes a generic positive constant independent of* ε_k*.*

3. Construction of difference scheme

The kth equation of (1.1) is as follows

$$-\varepsilon_k u_k'' + b_k(x)u_k' + c_k(x,u_1,u_2,\ldots,u_l) = f_k(x), \tag{3.1}$$

$$u_k(0) = u_k(1) = 0 \quad for \quad k = 1,2,\ldots,l,$$

On $\overline{w} = [0,1]$, we introduce a non uniform mesh

$$\overline{w}^h = \{0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1\} \quad , \quad h_i = x_{i+1} - x_i.$$

Let G_i be the Green's function for the operator $-\varepsilon_k \frac{d^2}{dx^2} + b_k(x_i)\frac{d}{dx}$ on $[x_{i-1}, x_{i+1}]$. In this case the exact solution of (3.1) is

$$u_k(x) = u_{k;i-1}\phi_{ki}^I(x) + u_{k;i+1}\phi_{ki}^{II}(x) + \int_{x_{i-1}}^{x_{i+1}} G_{ki}(x,s)\psi_k(s)ds, \tag{3.2}$$

where

$$\psi_k(s) = -c_k(s, u_1(s), \ldots .u_l(s)) + f_k(s),$$

and

$$G_{ki}(x,s) = \frac{1}{-\varepsilon_k w_{ki}(s)} \begin{cases} \phi_{ki}^I(s)\phi_{ki}^{II}(x) & x_{i-1} \leq x \leq s \leq x_{i+1}, \\ \phi_{ki}^I(x)\phi_{ki}^{II}(s) & x_{i-1} \leq s \leq x \leq x_{i+1}, \end{cases}$$

$$\phi_{ki}^I(x) = \frac{1-\exp(\frac{-b_{k;i}(x_{i+1}-x)}{\varepsilon_k})}{1-\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k})} \quad , \quad \phi_{ki}^{II}(x) = 1 - \phi_{ki}^I(x),$$

where in the above we have set $u_k(x_{i+1}) = u_{k;i+1}$, $u_k(x_{i-1}) = u_{k;i-1}$ and $b_k(x) = b_{k,i}$ for $x \in [x_{i-1}, x_{i+1}]$.

We have

$$\begin{aligned} u_k(x_i) &= u_{k;i-1}\phi^I_{ki}(x_i) + u_{k;i+1}\phi^{II}_{ki}(x_i) + \int_{x_{i-1}}^{x_i} G_{ki}(x_i,s)\psi_k(s)ds \\ &+ \int_{x_i}^{x_{i+1}} G_{ki}(x_i,s)\psi_k(s)ds, \end{aligned} \tag{3.3}$$

and

$$w_{ki}(s) = \phi^{II}_{ki}(s){\phi'}^{I}_{ki}(s) - \phi^I_{ki}(s){\phi'}^{II}_{i}(s) = \frac{-b_{ki}\exp(\frac{-b_{ki}(x_{i+1}-s)}{\varepsilon_k})}{\varepsilon_k(1-\exp(\frac{-b_{ki}(h_i+h_{i-1})}{\varepsilon_k}))}. \tag{3.4}$$

By (3.3) and (3.4) we can write

$$\begin{aligned} u_k(x_i) &= \frac{1}{1-\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k})}[(u_{k;i+1}-u_{k;i-1})\exp(\frac{-b_{k;i}h_i}{\varepsilon_k}) + u_{k;i-1} - \\ & u_{k;i+1}\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k}) + \frac{1-\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k})}{b_{k;i}} \times \\ & \int_{x_{i-1}}^{x_i}(1-\exp(\frac{-b_{k;i}(s-x_{i-1})}{\varepsilon_k}))\psi_k(s)ds + \frac{\exp(\frac{-b_{k;i}h_i}{\varepsilon_k}) - \exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k})}{b_{k;i}} \times \\ & \int_{x_i}^{x_i+1}[\exp(\frac{b_{k;i}(x_{i+1}-s)}{\varepsilon_k}) - 1]\psi_k(s)ds]. \end{aligned} \tag{3.5}$$

Now we approximate $\psi_k(s)$ and $f(s)$ for $s \in [x_{i-1}, x_{i+1}]$ by their values at x_i ($\psi_k(s) \simeq \psi_k(x_i)$, $f(s) \simeq f(x_i)$). Then we obtain

$$\begin{aligned} u_{k;i} &= u_k(x_i) \simeq \frac{1}{1-\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k})} \times \\ & [(u_{k;i+1}-u_{k;i-1})\exp(\frac{-b_{k;i}h_i}{\varepsilon_k}) + u_{k;i-1} - u_{k;i+1}\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k}) + \\ & \frac{\psi_k(x_i)}{b_{k;i}}[(1-\exp(\frac{-b_{k;i}h_i}{\varepsilon_k}))[h_{i-1} + \frac{\varepsilon_k}{b_{k;i}}(\exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k}) - 1)] + (\exp(\frac{-b_{k;i}h_i}{\varepsilon_k}) \\ & -\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k}))[\frac{-\varepsilon_k}{b_{k;i}}(1-\exp(\frac{b_{k;i}h_i}{\varepsilon_k})) - h_i]]] := v_{k;i}. \end{aligned} \tag{3.6}$$

Having the above approximation for (3.1), we introduce by the non standard finite difference method the following scheme

$$\begin{aligned} & -\varepsilon_k\alpha(h_{i-1},h_i)[v_{k;i-1} - 2v_{k;i} + v_{k;i+1}] + \\ & \beta(h_{i-1},h_i)b_{k;i}(v_{k;i+1} - v_{k;i-1}) - \psi_k(x_i) = 0, \end{aligned} \tag{3.7}$$

where

$$\psi_k(x_i) = -c_k(x_i, v_{1;i}, \dots, v_{l;i}) + f_k(x_i).$$

Now we find $\alpha(h)$ and $\beta(h)$, such that the scheme (3.7) is exact for (3.1). By substituting (3.6) in (3.7), we have

$$A_1(v_{k;i-1} - v_{k;i+1}) + B_1\psi_k(x_i) = 0, \tag{3.8}$$

for $i = 1, \dots, N-1$. where

$$A_1 = -\varepsilon_k\alpha(h_{i-1}, h_i) - \frac{2\varepsilon_k\alpha(h_{i-1}, h_i)(\exp(\frac{-b_{k;i}h_i}{\varepsilon_k}) - 1)}{1 - \exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k})} - \beta(h_{i-1}, h_i)b_{ki},$$

and

$$\begin{aligned} B_1 \;&=\; -1 + \frac{2\alpha(h_{i-1}, h_i)}{\varepsilon_k b_{k;i}(1 - \exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k}))}[h_{i-1}(1 - \exp(\frac{-b_{k;i}h_i}{\varepsilon_k})) + \\ &\quad h_i(\exp(\frac{-b_{k;i}(h_i + h_{i-1})}{\varepsilon_k}) - \exp(\frac{-b_{k;i}h_i}{\varepsilon_k}))], \end{aligned}$$

In (3.8) A_1 and B_1 must be zero. By setting A_1 and B_1 to zero we obtain

$$\alpha(h_{i-1}, h_i) = \frac{b_{k;i}[\exp(\frac{b_{k;i}h_i}{\varepsilon_k}) - \exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k})]}{2\varepsilon_k[h_{i-1}(\exp(\frac{b_{k;i}h_i}{\varepsilon_k}) - 1) + h_i(\exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k}) - 1)]},$$

and

$$\beta(h_{i-1}, h_i) = \frac{1}{2}\frac{\exp(\frac{b_{k;i}h_i}{\varepsilon_k}) - 2 + \exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k})}{h_{i-1}(\exp(\frac{b_{k;i}h_i}{\varepsilon_k}) - 1) + h_i(\exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k}) - 1)},$$

Remark 3.1. For uniform mesh, the scheme (3.7) becomes the IL'in scheme [10].

By appling $\alpha(h_{i-1}, h_i)$ and $\beta(h_{i-1}, h_i)$ in (3.7) we obtain

$$g_{ki}v_{k;i-1} + h_{ki}v_{k;i} + \gamma_{ki}v_{k;i+1} + c_k(x_i, v_{1;i}, \dots, v_{l;i}) - f_k(x_i) = 0, \tag{3.9}$$

$$for \quad k = 1, 2, \dots, l \quad and \quad i = 1, 2, \dots, N-1,$$

where

$$\begin{aligned} g_{ki} \;&=\; \frac{-b_{k;i}[\exp(\frac{b_{k;i}h_i}{\varepsilon_k}) - 1]}{h_{i-1}[\exp(\frac{b_{k;i}h_i}{\varepsilon_k}) - 1)] + h_i[\exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k}) - 1)]} \\ h_{ki} \;&=\; \frac{b_{k;i}[\exp(\frac{b_{k;i}h_i}{\varepsilon_k}) - \exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k})]}{h_{i-1}[\exp(\frac{b_{k;i}h_i}{\varepsilon_k}) - 1] + h_i[\exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k}) - 1]}, \end{aligned}$$

and

$$\gamma_{ki} = \frac{-b_{k;i}[1 - \exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k})]}{h_{i-1}[\exp(\frac{b_{k;i}h_i}{\varepsilon_k}) - 1] + h_i[\exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k}) - 1]},$$

we note that in (3.9), $g_{ki} + h_{ki} + \gamma_{ki} \geq 0$, for $k = 1, 2, \ldots, l$, $i = 1, 2, \ldots, N-1$ and the term $h_{i-1}(\exp(\frac{b_{k;i}h_i}{\varepsilon_k}) - 1) + h_i(\exp(\frac{-b_{k;i}h_{i-1}}{\varepsilon_k}) - 1)$ is positive, also g_{ki} and γ_{ki} are negative.

4. Uniform convergence of the scheme

Lemma 4.1. *Suppose A is a matrix such that $a_{ii} > 0$ and $a_{ij} \leq 0$ for $(i \neq j)$ and $i, j = 1, \ldots, n$. Also assume that $\sum_{k=1}^{n} a_{jk} \geq 0$ for $j = 1, \ldots, n$. Then for every arbitrary vector $\eta = (\eta_1, \ldots, \eta_n)^T$ we have*

$$\|\eta\|_{\infty,\omega} \leq \|A\eta\|_{\infty,\omega}.$$

Proof. Suppose for the element j of η, $\|\eta\|_{\infty,\omega} = |\eta_j|$. Without lose of generality, let $|\eta_j| = \eta_j$ (otherwise we consider $\|A\eta\| = \|-A\eta\|$).

$$(A\eta)_j = \sum_{k=1}^{n} a_{jk}\eta_k = a_{jj}\eta_j + \sum_{k=1,k\neq j}^{n} a_{jk}\eta_k \geq a_{jj}\eta_j + \sum_{k=1,k\neq j}^{n} a_{jk}|\eta_k|,$$

(since $a_{jk} \leq 0$ for $j \neq k$). Therefore

$$\begin{aligned}
(A\eta)_j - \eta_j &\geq a_{jj}\eta_j + \sum_{k=1,k\neq j}^{n} a_{jk}|\eta_k| - \eta_j \\
&\geq a_{jj}\eta_j + \sum_{k=1,k\neq j}^{n} a_{jk}|\eta_k| - \sum_{k=1}^{n} a_{jk}\eta_j \\
&= \sum_{k=1,k\neq j}^{n} a_{jk}(\eta_k - \eta_j) \geq 0.
\end{aligned}$$

So

$$(A\eta)_j \geq \eta_j > 0.$$

Hence

$$\|A\eta\|_{\infty,\omega} = \max_{k=1,\ldots,n} |(A\eta)_k| \geq |(A\eta)_j| \geq \eta_j = \|\eta\|_{\infty,\omega}.$$

□

Theorem 4.1. *The non linear difference scheme (3.9) on arbitrary meshes converges ε-uniformly to the solution of problem (3.1):*

$$\max_{0\leq i\leq N, 1\leq k\leq l} |u_k(x_i) - v_{k;i}| \leq Ch \quad , \quad h = \max_{0\leq i\leq N-1} h_i.$$

Proof. We now estimate the error

$$Z_{k;i} = u_k(x_i) - v_{k;i},\ 0 \leq i \leq N,$$

in the approximation of the problem (3.1) by the nonlinear scheme (3.9). now we subtract (3.7) from $2\varepsilon_k\alpha(h_{i-1}, h_i)$ times of (3.5), then by using the fact that

$$\frac{-2\varepsilon_k\alpha(h_{i-1}, h_i)}{1-\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k})}[\frac{1-\exp(-\frac{b_{k;i}h_i}{\varepsilon_k})}{b_{k;i}}\int_{x_{i-1}}^{x_i}[1-\exp(\frac{-b_{k;i}(s-x_{i-1})}{\varepsilon_k})]ds +$$
$$\frac{\exp(\frac{-b_{k;i}h_i}{\varepsilon_k})-\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k})}{b_{k;i}}\int_{x_i}^{x_i+1}[\exp(\frac{b_{k;i}(x_{i+1}-s)}{\varepsilon_k})-1]ds = -1,$$

we have

$$g_{ki}Z_{k;i-1} + h_{ki}Z_{k;i} + \gamma_{ki}Z_{k;i+1} - \frac{2\varepsilon_k\alpha(h_{i-1}, h_i)}{1-\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k})}[\frac{1-\exp(\frac{-b_{k;i}h_i}{\varepsilon_k})}{b_{k;i}} \times$$
$$\int_{x_{i-1}}^{x_i}[1-\exp(\frac{-b_{k;i}(s-x_{i-1})}{\varepsilon_k})][\psi_k(s)-\psi_k(x_i)]ds +$$
$$\frac{\exp(\frac{-b_{k;i}h_i}{\varepsilon_k})-\exp(\frac{-b_{k;i}(h_i+h_{i-1})}{\varepsilon_k})}{b_{k;i}}\int_{x_i}^{x_i+1}[\exp(\frac{b_{k;i}(x_{i+1}-s)}{\varepsilon_k})-1] \times$$
$$[\psi_k(s)-\psi_k(x_i)]ds] = 0. \tag{4.1}$$

We note that for $s \in [x_{i-1}, x_i]$ we have

$$c_k(s, u_1(s), \ldots, u_l(s)) = c_k(x_i, u_1(x_i), \ldots, u_l(x_i)) - \int_s^{x_i}\frac{dc_k}{dx}dx,$$

and for $s \in [x_i, x_{i+1}]$ we have

$$c_k(s, u_1(s), \ldots, u_l(s)) = c_k(x_i, u_1(x_i), \ldots, u_l(x_i)) + \int_{x_i}^{s}\frac{dc_k}{dx}dx,$$

and by the mean-value theorem for $s \in [x_{i-1}, x_i]$

$$c_k(x_i, u_1(x_i), \ldots, u_l(x_i)) = c_k(x_i, v_{1;i}, \ldots, v_{l;i}) + \sum_{j=1}^{l}\frac{\partial c_k}{\partial u_j}Z_{j;i},$$

so for $s \in [x_{i-1}, x_i]$

$$\psi_k(s) - \psi_k(x_i) = -\sum_{j=1}^{l} \frac{\partial c_k}{\partial u_j} Z_{j;i} + \int_s^{x_i} \frac{dc_k}{dx} dx,$$

and for $s \in [x_i, x_{i+1}]$

$$\psi_k(s) - \psi_k(x_i) = -\sum_{j=1}^{l} \frac{\partial c_k}{\partial u_j} Z_{j;i} - \int_{x_i}^{s} \frac{dc_k}{dx} dx.$$

Therefore (4.1) reduce to

$$\begin{aligned}
&g_{ki} Z_{k;i-1} + h_{ki} Z_{k;i} + \gamma_{ki} Z_{k;i+1} + \sum_{j=1}^{l} \frac{\partial c_k}{\partial u_j} Z_{j;i} = \\
&-\frac{2\varepsilon_k \alpha(h_{i-1}, h_i)}{1 - \exp(\frac{-b_{k;i}(h_i + h_{i-1})}{\varepsilon_k})} [\frac{1 - \exp(\frac{-b_{k;i} h_i}{\varepsilon_k})}{b_{k;i}} \int_{x_{i-1}}^{x_i} [-1 + \exp(\frac{-b_{k;i}(s - x_{i-1})}{\varepsilon_k})] \times \\
&(\int_s^{x_i} \frac{dc_k}{dx} dx) ds + \frac{\exp(\frac{-b_{k;i} h_i}{\varepsilon_k}) - \exp(\frac{-b_{k;i}(h_i + h_{i-1})}{\varepsilon_k})}{b_{k;i}} \times \\
&\int_{x_i}^{x_{i+1}} [\exp(\frac{b_{k;i}(x_{i+1} - s)}{\varepsilon_k}) - 1] (\int_{x_i}^{s} \frac{dc_k}{dx} dx) ds] := \Psi(s).
\end{aligned} \tag{4.2}$$

By lemma 2.2 and the fact that c_k is sufficiently smooth we have

$$|\frac{dc_k}{dx}| \le c(1 + \varepsilon_k^{-1} \exp(\frac{-\beta_k(1-x)}{\varepsilon_k})).$$

By lemma 4.1 we have

$$\|Z\|_\infty \le \|\Psi\|_\infty.$$

By doing some algebra, we can show that $\|\psi(s)\| \le Ch$. Thus

$$\|Z\|_\infty \le \|\Psi\|_\infty \le Ch.$$

□

5. Monotone iterative method

In this section, we construct an iterative method for solving the nonlinear difference scheme (3.9) which possesses the monotone convergence. This method is based on the approach used in [5].

Additionally, we assume that $c(x, u_1(x), \dots, u_l(x))$ in (3.1) satisfies in (1.3).
We introduce the linear version of (3.9) as follows

$$g_{ki}W_{k;i-1} + h_{ki}W_{k;i} + \gamma_{ki}W_{k;i+1} + \sum_{j=1}^{l} c_{kj;i}W_{j;i} + f_k(x_i) = 0, \tag{5.1}$$

for $k = 1, 2, \dots, l \quad and \quad i = 1, 2, \dots, N-1$. In (5.1), suppose $c_{pp} > 0$, $c_{pq} \le 0 \quad (p \ne q)$ and $\sum_{q=1}^{l} c_{pq} \ge 0$ for $p, q = 1, 2, \dots, l$.
The iterative method is constructed as follows. Choose an initial mesh function $V_k^{(0)} = (V_{k;0}^{(0)}, V_{k;1}^{(0)}, \dots, V_{k;N}^{(0)})$ satisfying the boundary conditions $V_{k;0}^{(0)} = V_{k;N}^{(0)} = 0$. The sequence $\{V_k^{(n)}\}_{n\ge1}$, for $k = 1, \dots, l$, is defined by the following recurrence formula:

$$g_{ki}Z_{k;i-1}^{(n)} + (h_{ki} + c_{kk,i}^{*})Z_{k;i}^{(n)} + \gamma_{ki}Z_{k;i+1}^{(n)} + \sum_{j=1, j\ne k}^{l} c_{kj}^{*}Z_{j;i}^{(n)} = -R_{ki}^{h}(x, V_k^{(n-1)}), \tag{5.2}$$

$$Z_{k;0}^{(n)} = Z_{k;N}^{(n)} = 0 \quad 1 \le i \le N-1 \quad , \quad k = 1, 2, \dots, l,$$
$$V_{k;i}^{(n)} = V_{k;i}^{(n-1)} + Z_{k;i}^{(n)} \quad 0 \le i \le N,$$

$$R_{ki}^{h}(x, V_k^{(n-1)}) = g_{ki}V_{k;i-1}^{(n-1)} + h_{ki}V_{k;i}^{(n-1)} + \gamma_{ki}V_{k;i+1}^{(n-1)} + c_k(x_i, V_{1;i}^{(n-1)}, \dots, V_{l;i}^{(n-1)}) + f_k(x_i),$$

where $R_{ki}^{h}(x, V_k^{(n-1)})$ is the residual of the difference scheme (3.9) on $V_k^{(n-1)}$.
We say that $\overline{V}_k(x)$ is an upper solution of (3.9) if it satisfies the following inequality.

$$g_{ki}\overline{V}_{k;i-1} + h_{ki}\overline{V}_{k;i} + \gamma_{ki}\overline{V}_{k;i+1} + c_k(x_i, \overline{V}_{1;i}, \dots, \overline{V}_{l;i}) + f_k(x_i) \ge 0.$$

Similarly, $\underline{V}_k(x)$ is called a lower solution if it satisfies

$$g_{ki}\underline{V}_{k;i-1} + h_{ki}\underline{V}_{k;i} + \gamma_{ki}\underline{V}_{k;i+1} + c_k(x_i, \underline{V}_{1;i}, \dots, \underline{V}_{l;i}) + f_k(x_i) \le 0.$$

Upper and lower solutions satisfy the inequality

$$\underline{V}_{k;i}(x) \le \overline{V}_{k;i}(x),$$

for $i = 0, 1, \dots, N \quad , \quad k = 1, 2, \dots, l$ and $x \in \overline{w}^h$.

Theorem 5.1. *Let $\overline{V}_k^{(0)}$ and $\underline{V}_k^{(0)}$ be upper and lower solution of (3.9) and let $c_k(x, V_1(x), \dots, V_l(x))$ satisfies (1.3). Then the upper sequence $\{\overline{V}^{(n)}\}_{n\ge1}$ generated by (5.2) converges monotonically from above to the unique solution V_k of (3.9), the lower sequence $\{\underline{V}^{(n)}\}$ generated by (5.2) converges monotonically from below to V_k.*

$$\underline{V}_k^{(0)} \le \underline{V}_k^{(n)} \le \underline{V}_k^{(n+1)} \le \underline{V}_k \le \overline{V}_k^{(n+1)} \le \overline{V}_k^{(n)} \le \overline{V}^{(0)}$$

on $\overline{w}^h$, and the sequences converge at the linear rate

$$q = \max_{k=1}^{l} \frac{\sum_{j=1}^{l} c^*_{kj,i} - \sum_{j=1}^{l} c_{*kj,i}}{\sum_{j=1}^{l} c^*_{kj,i}}.$$

Proof. We consider only the case of the upper sequence. If $\overline{V}_k^{(0)}$ is an upper solution then from (3.9) we conclude that

$$R^h_{ki}(x, V_k^{(0)}) = g_{ki}\overline{V}^{(0)}_{k;i-1} + h_{ki}\overline{V}^{(0)}_{k;i} + \gamma_{ki}\overline{V}^{(0)}_{k;i+1} + c_k(x_i, \overline{V}^{(0)}_{1;i}, \ldots, \overline{V}^{(0)}_{l;i}) + f_k(x_i) \geq, 0$$

from (5.2) we have

$$-R^h_{ki}(x, V_k^{(0)}) = g_{ki}Z^{(1)}_{k;i-1} + (g_{ki} + c^*_{kk;i})Z^{(1)}_{k;i} + \gamma_{ki}Z^{(1)}_{k;i+1} + \sum_{j=1}^{l} c^*_{kj;i}Z_{j;i} \leq 0.$$

By lemma 2.1 we have $Z^{(1)}_{k;i} \leq 0$, therefore

$$V^{(1)}_{k;i} = Z^{(1)}_{k;i} + V^{(0)}_{k;i} \leq V^{(0)}_{k;i} \quad , \quad Z^{(1)}_{k;0} = Z^{(1)}_{k;N} = 0.$$

To show that $V_k^{(1)}$ is upper solution of (3.9) we must prove that $R^h_{ki}(V_k^{(1)}) \geq 0$.
Using the mean-value theorem and the equation for $Z_k^{(1)}$, we represent $R^h_k(x, V_k^{(1)})$ in the form

$$R^h_{ki}(x, V_k^{(1)}) = \sum_{j=1}^{l}(-c^*_{kj;i} + \frac{\partial c_k}{\partial V_j})Z^{(1)}_{j;i} \geq 0, \tag{5.3}$$

from (5.3) we conclude that $\overline{V}_k^{(1)}$ is an upper solution. By induction we obtain that $Z^{(n)}(x) \leq 0, x \in \overline{w}^h$ and $V^{(n+1)}_{k;i} \leq V^{(n)}_{k;i} \quad n = 1, 2, \ldots,$ and prove that $\{\overline{V}_k^{(n)}\}$ is a monotonically decreasing sequence of upper solutions. We now prove that the monotone sequence $\{\overline{V}_k^{(n)}\}$ converges to the solution of (3.9).
Similar to (5.3), we obtain

$$R^h(x, \overline{V}_k^{(n)}) = \sum_{j=1}^{l}(-c^*_{kj;i} + \frac{\partial c_k}{\partial V_j})Z^{(n)}_{j;i} \geq 0,$$

therefore

$$g_{ki}Z^{(n+1)}_{k;i-1} + (h_{ki} + c^*_{kk;i})Z^{(n+1)}_{k;i} + \gamma_{ki}Z^{(n+1)}_{k;i+1} + \sum_{j=1, j\neq k}^{l} c^*_{kj;i}Z^{(n+1)}_{j;i} =$$

$$-R^h_{ki}(x, V_k^{(n)}) = \sum_{j=1}^{l}(c^*_{kj;i} - \frac{\partial c_k}{\partial V_j})Z^{(n)}_{j;i} \leq 0. \tag{5.4}$$

We take the absolute values of both side of (5.4),to obtain

$$-g_{ki}Z^{(n+1)}_{k;i-1}-(h_{ki}+c^*_{kk;i})Z^{(n+1)}_{k;i}-\gamma_{ki}Z^{(n+1)}_{k;i+1}-\sum_{j=1,j\neq k}^{l}c^*_{kj;i}Z^{(n+1)}_{j;i}=$$

$$-\sum_{j=1}^{l}(c^*_{kj;i}-\frac{\partial c_k}{\partial V_j})Z^{(n)}_{j;i}. \tag{5.5}$$

Since $Z^{(n+1)}_{k;i}\leq 0$, (5.5) reduces to

$$g_{ki}|Z^{(n+1)}_{k;i-1}|+(h_{ki}+c^*_{kk;i})|Z^{(n+1)}_{k;i}|+\gamma_{ki}|Z^{(n+1)}_{k;i+1}|+$$

$$\sum_{j=1,j\neq k}^{l}c^*_{kj;i}|Z^{(n+1)}_{j;i}|=\sum_{j=1}^{l}(c^*_{kj;i}-\frac{\partial c_k}{\partial V_j})|Z^{(n)}_{j;i}|. \tag{5.6}$$

By definition $\|Z\|_\infty=\max_{1\leq k\leq l,1\leq i\leq N-1}|Z_{k;i}|$, and the fact that $g_{ki}<0$ and $\gamma_{ki}<0$, we have

$$g_{ki}\|Z^{(n+1)}\|_\infty+(h_{ki}+c^*_{kk;i})\|Z^{(n+1)}\|_\infty+\gamma_{ki}\|Z^{(n+1)}\|_\infty+$$

$$\sum_{j=1,j\neq k}^{l}c^*_{kj;i}\|Z^{(n+1)}\|_\infty\leq\sum_{j=1}^{l}(c^*_{kj;i}-\frac{\partial c_k}{\partial V_j})|Z^{(n)}_{j;i}|\leq$$

$$\sum_{j=1}^{l}(c^*_{kj;i}-\frac{\partial c_k}{\partial V_j})\|Z^{(n)}\|_\infty\leq\sum_{j=1}^{l}(c^*_{kj;i}-c_{*kj;i})\|Z^{(n)}\|_\infty,$$

therefore

$$(g_{ki}+h_{ki}+c^*_{kk;i}+\gamma_{ki}+\sum_{j=1}^{l}c^*_{kj;i})\|Z^{(n+1)}\|_\infty\leq\sum_{j=1}^{l}(c^*_{kj;i}-c_{*kj;i})\|Z^{(n)}\|_\infty.$$

Since $g_{ki}+h_{ki}+\gamma_{ki}\geq 0$, we have

$$\|Z^{(n+1)}\|_\infty|\leq\frac{\sum_{j=1}^{l}c^*_{kj;i}-\sum_{j=1}^{l}c_{*kj;i}}{\sum_{j=1}^{l}c^*_{kj;i}}\|Z^{(n)}\|_\infty,$$

so

$$\|Z^{(n+1)}\|_\infty\leq q^n\|Z^{(1)}\|_\infty, \tag{5.7}$$

where

$$q=\max_{k=1}^{l}\frac{\sum_{j=1}^{l}c^*_{kj;i}-\sum_{j=1}^{l}c_{*kj;i}}{\sum_{j=1}^{l}c^*_{kj;i}}<1.$$

This proves convergence of the upper sequence at the linear rate q. We have from (5.7) and (5.2) that the mesh function $V_k(x)$ defined by

$$\lim_{n\to\infty} \overline{V}_k^{(n)}(x) = V_k(x) \quad x \in \overline{w}^h$$

is an exact solution to (3.9). The uniqueness of the solution to (3.9) follows from lemma 4.1. Indeed, if by contradiction, we assume that there exist two solutions $V^{(1)}$ and $V^{(2)}$ to (3.9), then by the mean-value theorem, the difference $\delta V = V^{(1)} - V^{(2)}$ satisfies the difference problem

$$g_{ki}\delta V_{k;i-1} + h_{ki}\delta V_{k;i} + \gamma_{ki}\delta V_{k;i+1} + \sum_{j=1}^{l} \frac{\partial c_k}{\partial V_j}\delta V_{j;i} = 0, \tag{5.8}$$

$$\delta V(0) = \delta V(1) = 0.$$

By lemma 4.1, $\|\delta V\|_\infty \leq 0$, therefore $V^{(1)} = V^{(2)}$. □

6. Numerical experiments

We solve the nonlinear difference scheme (3.9) on uniform meshes by the monotone iterative method (5.2). The stopping criterion is

$$\max_{x\varepsilon\varpi} |V^n(x) - V^{n-1}(x)| \leq \delta,$$

where δ is the required accuracy. If at step $n = n^*$ the stopping criterion is satisfied, then $V(x) = V^{n^*}(x), x \in \omega^h$, where $V(x)$ is the corresponding numerical solution. In the absence of an exact solution for test problems, for fixed value of ε, the nonlinear difference scheme (3.9) with $N = 2048$ is solved by the monotone iterative method (5.2) with the stopping criterion $\delta = 10^{-5}$. This generates a reference solution $V_{ref}(x)$. The basic feature of monotone convergence of the upper and lower sequences is observed in all the numerical experiments. In fact, the monotone property of the sequences holds at every mesh point in the domain, of course, this is expected from the analytical considerations.

Example 1. Consider the following test problem

$$\begin{aligned} -\varepsilon_1 u_1'' + 3u_1' + (u_1 + \frac{1}{3}u_1^3) - u_2 &= f_1(x) \qquad u_1(0) = u_1(1) = 0, \\ -\varepsilon_2 u_2'' + u_2' - u_1 + 2u_2 + \frac{1}{5}u_2^5 &= f_2(x) \qquad u_2(0) = u_2(1) = 0. \end{aligned}$$

N	$\varepsilon_1 = 10^{-1}$	$\varepsilon_2 = 10^{-1}$	$\varepsilon_1 = 10^{-2}$	$\varepsilon_2 = 10^{-2}$	$\varepsilon_1 = 10^{-3}$	$\varepsilon_2 = 10^{-3}$	$\varepsilon_1 = 10^{-4}$	$\varepsilon_2 = 10^{-4}$
32	1.771e-4	2.999e-4	6.506e-4	0.0031	5.392e-4	0.0048	5.392e-4	0.0048
64	0.453e-4	0.75e-4	3.248e-4	0.0009	2.794e-4	0.0024	2.793e-4	0.0024
128	0.114e-4	0.189e-4	1.114e-4	0.0002	1.452e-4	0.0012	1.421e-4	0.0012
256	0.029e-4	0.047e-4	0.307e-4	0.0001	0.821e-4	0.0005	0.717e-4	0.0006
512	0.007e-4	0.012e-4	0.079e-4	0.0000	0.438e-4	0.0001	0.360e-4	0.0003

TABLE 1. Maximal approximate error $\overline{E}_{N,\varepsilon}$ for the monotone iterative method (5.2) applied to the test problem 1.

N	$\bar{\alpha}_{N\varepsilon_1}$	$\bar{\alpha}_{N\varepsilon_2}$	$\bar{\alpha}^*_{N\varepsilon_1}$	$\bar{\alpha}^*_{N\varepsilon_2}$
32	0.9490	0.9713	1.0022	1
64	0.9752	0.9856	1.1615	1
128	0.9876	0.9927	0.8226	1
256	0.9936	0.9965	0.9065	1
512	-	-	-	-

TABLE 2. The numerical order of convergence $\bar{\alpha}_{N\varepsilon}$ for $\varepsilon_1 = \varepsilon_2 = 10^-4$, and the uniform numerical order of convergence $\bar{\alpha}^*_{N\varepsilon}$ for all ε_1 and ε_2 in Table1, applied to the test problem 1.

In this example, $c(x, u_1, u_2) = \begin{pmatrix} u_1 + \frac{1}{3}u_1^3 & -u_2 \\ -u_1 & 2u_2 + \frac{1}{5}u_2^5 \end{pmatrix}$, $\frac{\partial c}{\partial u} = \begin{pmatrix} 1 + u_1^2 & -1 \\ -1 & 2 + u_2^4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, $f_1(x) = 1$ and $f_2(x) = 2$. We consider $C_* = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and by lemma 2.2 there is C^* such that $C_* \leq \frac{\partial c}{\partial u} \leq C^*$ for $x \in \overline{w}^h = [0, 1]$. In our numerical experiments, the lower solution $\underline{V}^{(0)}(x) = 0$ for $x \in (0, 1)$ and $\underline{V}^{(0)}(0) = \underline{V}^{(0)}(1) = 0$. In Table 1 for various values of ε and N, we present the maximal approximate error $\overline{E}_{N,\varepsilon} = \max_{x \in \overline{w}_N^h} E_{N,\varepsilon}(x)$, $E_{N,\varepsilon}(x) \equiv |V_{N,\varepsilon}(x) - V_{ref,\varepsilon}(x)|$ where $V_{N,\varepsilon}(x)$ is the numerical solution of the nonlinear difference (3.9) by the monotone iterative method (5.2).

Fig. 1. shows for very small ε, the error is independent of ε and decreases with N, that is the nonlinear difference scheme by the monotone iterative method converges $\varepsilon-$uniformly.

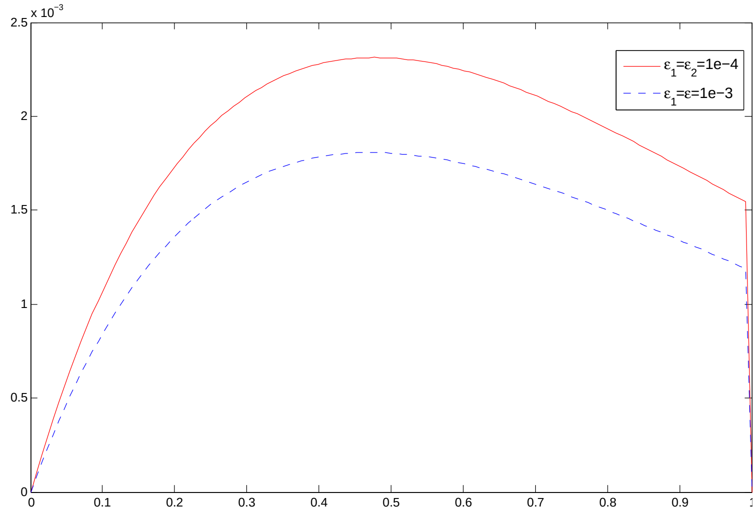

FIGURE 1. $E_{N,\varepsilon}$ with $N = 128$ and $\varepsilon_1 = \varepsilon_2 = 0.001$ and $\varepsilon_1 = \varepsilon_2 = 0.0001$ for $u_2(x)$ of the test problem 1.

The numerical order of convergence $\overline{\alpha}_{N,\varepsilon}$ and the uniform numerical order of convergence $\overline{\alpha}_N^*$ are calculated as in [7].

$$\overline{R}_{N,\varepsilon_k} = \max_{x\in\varpi_N^h} |V_N(x;\varepsilon_k) - V_{2N}(x;\varepsilon_k)|, \quad \overline{R}_N^* = \max_{\varepsilon_k} \overline{R}_{N,\varepsilon},$$

$$\overline{\alpha}_{N,\varepsilon_k} = \log_2(\frac{\overline{R}_{N,\varepsilon_k}}{\overline{R}_{2N,\varepsilon_k}}), \quad \overline{\alpha}_N^* = (\log_2 \frac{\overline{R}_N^*}{\overline{R}_{2N}^*}),$$

for $k = 1,\dots,l$, and are close to one (Table 2). This confirms the theoretical result in Theorem 4.1.

Example 2. Consider the following test problem:

$$-\varepsilon u'' + b(x)u' + c(x,u) + f(x) = 0, \quad u(0) = u(1) = 0,$$

$$c(x,u) = 1 - \exp(-u), \quad b(x) = 1, \quad f(x) = \begin{cases} 1, & x \le 0.5, \\ 0.5, & x > 0.5, \end{cases}$$

Consider the lower solution $\underline{V_0}(x) = 0$, $x \in \overline{\omega}^h$ to (3.9). We conclude that $c_* = \min_{0\le u\le 1} c_u = e^{-1}$, $c^* = \max_{0\le u\le 1} c_u = 1$, where c_* and c^* are defined in (3.1). In Table 3, the maximal approximate error is presented for various value of ε and N. The numerical order of convergence $\overline{\alpha}_{N,\varepsilon}$ and the uniform numerical order of convergence $\overline{\alpha}_N^*$ are close to one, which confirms the theoretical result in theorem 4.1. The approximate error $E_{N,\varepsilon}$ with

N	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.001$	$\varepsilon = 0.0001$
32	0.0062	0.0022	0.0038	0.0041
64	0.0032	0.0022	0.0018	0.0020
128	0.0016	0.0014	0.0008	0.0010
256	0.0007	0.0007	0.0003	0.0005
512	0.0003	0.0003	0.0002	0.0002
1024	0.0001	0.0001	0.0001	0.0001

TABLE 3. Maximal approximate error $\overline{E}_{N,\varepsilon}$ for the monotone iterative method (5.2) applied to the test problem 2.

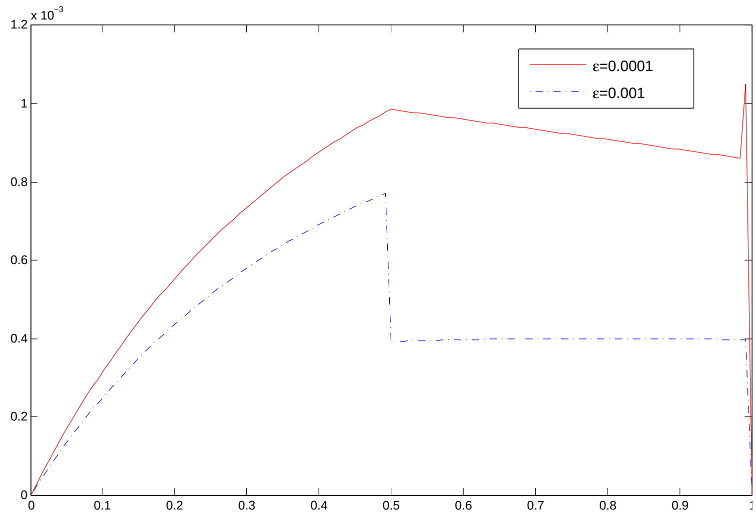

FIGURE 2. $E_{N,\varepsilon}$ with $N = 128$ and $\varepsilon = 0.001$ and $\varepsilon = 0.0001$ for the test problem 2.

$N = 128$ and $\varepsilon = 10^{-3}, 10^{-4}$ is depicted in Fig. 2. The maximum of the approximate error is attained in the boundary layer at $x = 1$.

REFERENCES

1. G.M. Amiraliyev, The convergence of a finite difference method on layer-adapted mesh for a singularly perturbed system, Appl. Math. Comput. **162** (2005) 1023-1024.
2. V. B. Andreev, The Green's function and a priori estimates of solution of monotone three point singularly perturbed finite-difference schemes, Differ. Equ. **37** (2001) 923-933.

3. S. Bellew, E. O'Riordan, A parameter-robust numerical method for a system of two singularly perturbed convection-diffusion equations, Appl. Numer. Math. **51** (2004) 171-186.
4. I. Boglaev, Sophie pack, A uniformy convergent method on arbitrary meshes for a semilinear convection-diffusion problem with discontinuous data, International Journal of numerical analysis and modeling., 5 (2008), 24-39
5. I. Boglaev, Uniform numerical methods on orbitrary meshes for singularly perturbed problems with discontinous data' Appl. Math. Comput., 154 (2004), 815-833. Sophie pack, A uniformy convergent
6. Z. Cen, Parameter-uniform finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations, Int. J. Comput. Math. **82** (2005) 177-192.
7. P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan and G.I. Shiskin, Robust Computational Techniques for Boundary Layers, CRC Press, 2000.
8. P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan and G.I. Shiskin, Singularly perturbed convection-diffusion problems with boundary and weak layers, J. Comput. Appl. Math., 166 (2004), 133-151.
9. J.L. Gracia, F.J. Lisbona, A uniformly convergent scheme for a system of reaction-diffusion equations, J. Comput. Appl. Math. **206** (2007) 1-16.
10. A. M. Il'in, A difference scheme for a differential equation with a small parameter affecting the highest derivative (in Russian). Mat. Zametki. 6 (1969) 237-248.
11. T. Linss, Dresden, Analysis of an upwind finite-difference scheme for a system of coupled singularly perturbed convection-diffusion equations, computing. **79** (2007) 23-32.
12. T. Linss, Analysis of a system of singularly perturbed covection-diffusion equations with strong coupling, SIAM J. Numer. Anal. **47** (2009) 1847-1862.
13. T. Linss, N. Madden, Accurate solution of a system of coupled singularly perturbed reaction-diffusion equations, Computing. **73** (2004) 121-133.
14. N. Madden, M. Stynes, A uniformly convergent numerical method for a coupled system of two singularly perturbed linear reaction-diffusion problems, IMA J. Numer. Anal. **23** (4) (2003) 627-644.
15. H.G. Roos, M. Stynes, and L. Tobiska, Robust methods for singularly perturbed differential equations, 2nd ed.,Springer Ser. Comput. Math. **24**, Springer, Berlin, 2008.

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