AMO-Advanced Modeling and Optimization, Volume 13, Number 3, 2011

On Generalized Fuzzy Numbers

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Abstract

Our paper analyzes some new lines on analythical concepts, as the so-called Generalized Fuzzy Number, also introducing Triangular Numbers, Trapezoidal Numbers, and so on, on Universes of Discourse, jointly with some new ideas about the Generalized Fuzzy Complex Numbers.

Keywords: Fuzzy Sets, Fuzzy Numbers, Measure Theory, Fuzzy Measures.

Mathematics Subject Classification: 68R10, 68R05, 05C78, 78M35, 94A17.

1. Generalized Fuzzy Numbers

A *GFN* (*Generalized Fuzzy Number,* by acronym), *F,* will be any fuzzy subset [1, 2, 3] of the real line whose membership function, μ_F, verifying:

1) $\mu_F(x) = 0$, if $-\infty < x \leq a$

2) $\mu_F(x) = L(x)$ is strictly increasing on $[a, b]$

3) $\mu_F(x) = w$, if $b \leq x \leq c$

4) $\mu_F(x) = R(x)$ is strictly decreasing on $[c, d]$

5) $\mu_F(x) = 0$, if $d \leq x < +\infty$

The more usual way [9, 10] to denote a GFN may be

$$A = (a, b, c, d; w)$$

In particular case when $w = 1$, we can express the GFN by

$$A = (a, b, c, d)_{LR}$$

Obviously, when the functions *L(x)* and *R(x)* corresponds to straight lines, we have a *Trapezoidal Fuzzy Number.*

2. Complex Fuzzy Numbers

Let U be the universe of discourse [1, 4, 5]. Then, we define a *Complex Fuzzy Set* (*CFS,* by acronym), and denoted as *C,* through its membership function,

[1] **AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

$$\mu_C(x) = r_C(x) \ \ \exp\{i\ \omega_C(x)\}$$

where i represent the imaginary unit, being r_C and ω_C both real-valued functions, with an important restriction on r_C :

$$0 \leq r_C(x) \leq 1$$

Hence, a CFS may be represented as an ordered collection of pairs of this type:

$$C = \{x \mid \mu_C(x)\}_{x\in U} = \{(x,\ \mu_C(x))\}_{x\in U}$$

We can consider the precedent membership function, $\mu_C(x)$, as composed by two factors [2, 4]: *membership amplitude,* $r_C(x)$, and *membership phase,* $\omega_C(x)$.

As a particular case, we may consider the case of C with a null membership function, $\mu_C(x) = 0$; for instance on the case where either $r_C(x)$, or $r_C(x)$ and $\omega_C(x) = 0$; therefore, null phase and amplitude.

We make a more detailed analysis of membership phase component, wrt the fuzzy operations, union, intersection, and so on.

Let A and B two CFS.
Then,we can define

$$\mu_{A\cup B}(x) = \ \ [r_A(x) \blacktriangleright r_B(x)] \ \ \exp\{i\ \omega_{A\cup B}(x)\}$$

being ▶ here some *T-conorm operator.*

And similarly (but being different) we can also define

$$\mu_{A\cap B}(x) = \ \ [r_A(x) \blacktriangleleft r_B(x)] \ \ \exp\{i\ \omega_{A\cap B}(x)\}$$

being ◀ in this case some *T-norm operator.*

It remains until now without definition their respective phases,

$$\omega_{A\cup B}(x)\,,\ \ and\ \omega_{A\cap B}(x)$$

3. The Lattice of Fuzzy Numbers

To expose the fundamental operations between fuzzy numbers, we firstly analyze the so-called *Interval Operations* [4, 6],

- *Addition,*

$$[a,b]\,(+)\,[c,d] = [a+c, b+d]$$

- *Difference,*

$$[a, b]\,(-)\,[c, d] = [a - d, b - c]$$

- *Product,*

$$[a, b]\,(\cdot)\,[c, d] = [ac \wedge ad \wedge bc \wedge bd,\ ac \vee ad \vee bc \vee bd]$$

- *Division,*

$$[a, b]\,(/)\,[c, d] = [a/c \wedge a/d \wedge b/c \wedge b/d,\ a/c \vee a/d \vee b/c \vee b/d]$$

on the last case, defined in this way except when

$$c = d = 0$$

Two examples may be either

$$[1, 2]\,(+)\,[3, 4] = [1 + 3, 2 + 4] = [4, 6]$$

or

$$[0, 1]\,(+)\,[-3, 5] = [0 + (-3)\,, 1 + 5] = [-3, 6]$$

The substraction of the same both fuzzy numbers will be either

$$[1, 2]\,(-)\,[3, 4] = [1 - 4, 2 - 3] = [-3, -1]$$

or

$$[0, 1]\,(-)\,[-3, 5] = [0 - 5, 1 - (-3)] = [0, 4]$$

In the first example for division, it produces

$$\begin{gathered}[1, 2]\,(\cdot)\,[3, 4] = [1 \cdot 3 \wedge 1 \cdot 4 \wedge 2 \cdot 3 \wedge 2 \cdot 4,\ 1 \cdot 3 \vee 1 \cdot 4 \vee 2 \cdot 3 \vee 2 \cdot 4] = \\ = [3 \wedge 4 \wedge 6 \wedge 8,\ 3 \vee 4 \vee 6 \vee 8] = [3, 8]\end{gathered}$$

And in the case of division between both fuzzy numbers of the first example, we have

$$[1, 2]\,(/)\,[3, 4] = [1/3 \wedge 1/4 \wedge 2/3 \wedge 2/4,\ 1/3 \vee 1/4 \vee 2/3 \vee 2/4] = [1/4, 2/3]$$

This essential arithmetics may be also expressed by interval operations of $\alpha - level\ sets$.

Let $(*) \in \{(+), (-), (\cdot), (/)\}$ any of such operations, with the restriction that $0 \notin B^{\alpha}$, on the last case, and being also $0 < \alpha \leq 1$.

Then, we obtain this fuzzy number

$$A\,(*)\,B = \cup_{\alpha \in (0,1]} \left[A\,(*)\,B\right]^{\alpha}$$

The generalization is possible, because we dispose of the *Extension Principle,* for any arithmetic operation, $(*)$.

Because this may be expressed by this crucial result

$$\left[A\,(*)\,B\right](z) = \sup_{z=x(*)y} \{\min\left[A\,(x)\,,\ B\,(y)\right]\}$$

Recall that a *Lattice* is a poset (partially ordered set) with an ordering relation. We dispose in this case of

- *Meet* (g. l. b., or greatest lower bound),

and

- *Join* (l. u. b., or least upper bound)

operations.

So, we can describe the *Lattice of Fuzzy Numbers* by

$$MIN\,(A, B) = \sup_{z=\min(x,y)} \{\min\left[A\,(x)\,,\ B\,(y)\right]\} = MEET\,(A, B)$$
$$and$$
$$MAX\,(A, B) = \sup_{z=\max(x,y)} \{\min\left[A\,(x)\,,\ B\,(y)\right]\} = JOIN\,(A, B)$$

Also that a distributive lattice, in this case, signifies that

$$MIN\left[A,\ MAX\,(B, C)\right] = MAX\left[MIN\,(A, B)\,,\ MIN\,(A, C)\right]$$
$$and$$
$$MAX\left[A,\ MIN\,(B, C)\right] = MIN\left[MAX\,(A, B)\,,\ MAX\,(A, C)\right]$$

Hence,

$$< \mathbf{F},\ MIN,\ MAX >$$

is a *distributive lattice,* being $\mathbf{F}$ the family, or collection, of all fuzzy sets.

Conclusion

Expressing by adequate Generalized Fuzzy Numbers will be a very inspired and useful idea. In fact, their applications extends to many different and promising fields, as may be on Fuzzy Logic, Measure Theory, Automata Theory, and so on.

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