Classifying Fuzzy Numbers

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Abstract

Our paper analyzes some new lines to advance on analythical concepts, as can be the so-called Fuzzy Number, introducing Interval Numbers, Triangular Numbers, Trapezoidal Numbers, and so on, on Universes of Discourse, jointly with new ideas about Fuzzy Complex Numbers. It will be very necessary to analyze the inner relationships with some other fuzzy numbers, giving so place to very interesting applications.

Keywords: Fuzzy Sets, Fuzzy Numbers, Measure Theory, Fuzzy Measures.

Mathematics Subject Classification: 68R10, 68R05, 05C78, 78M35, 94A17.

1. On Fuzzy Logic

The introduction of concepts and methods of Fuzzy Logic, where the idea of sets, relations and so on, must be modified in the sense of covering adequately the indetermination or imprecision of the real world.

We define the *"world"* as a complete and coherent description of how things are or how they could have been. In the problems related with this *"real world"*, which is only one of the *"possible worlds"*, the Monotonic Logic does not work with frequency. Such type of Logic is the classical in formal worlds, such as in Mathematics. But it is necessary to provide our investigations with a mathematical construct that can express all the "grey tones", not te classical representation of real world as either black or white, all or nothing, but as in the common and natural reasoning, through progressive gradation.

Logic analyzes the notion of *consequence.* So, it deals with Propositions or Set of Propositions, and their mutual relationships. Formal Logic [1, 2] attempt to represent all this by means of well-defined logical calculi. Some calculi differ in their definitions of sentences and their respective notion of consequence. So, for instance, on Propositional Logic, on Predicate Logic, on Modal Logic, and so on.

It is very usual that a logical calculus has two notions of consequence: *syntactical,* i.e. based on a notion of proof; and *semantical,* i.e. based on a notion

[1] **AMO - Advanced Modeling and Optimization. ISSN: 1841-4311**

of truth. Then, there are some natural questions, as soundness (so, does probability implies truth?), and completeness (so, does truth implies probability?).

In Medicine, a fuzzy proposition may be true to some degree, and this degree may be evolving with time. So, the sentence "the patient is old" is true in some degree: the older the age of the patient, the truer the sentence. Standard examples of fuzzy propositions may include linguistic variables, such as the related with age, being possible values: young, medium, old, with intermediate degrees, by fuzzy modifiers, as almost, something, enough, very, etc.

Fuzzy Logic has two different meanings [2-6]: *wide FL,* and *narrow FL.* In this last sense, *narrow Fuzzy Logic, nFL* by acronym, is a logical system that attempts to formalize the approximate reasoning. So, it will be an extension of Multi-Valued Logic. Therefore, this nFL has a much wider range of applications than traditional logical systems, because the apparition of new concepts, such as canonical form, Extension Principle, fuzzy IF-THEN rule, or Compositional Rule, etc.

Instead of this, if we consider *Fuzzy Logic in its widest sense, FLw* by acronym, it will be a synonimous with Fuzzy Set Theory (FST, by acronym). Indeed, it is the theory of classes with unsharp bundaries. Therefore, FST is much broader than FLn, including this latter as one of their branches. So, in the broad sense, everything dealing with fuzziness may be called a Fuzzy Logic. But in the narrow sense, the base of Fuzzy Logic will be the formal calculus of Multi-Valued Logic.

2. Fuzzy Numbers

A *fuzzy set, C,* on the real line is defined to be a collection of ordered pairs [2, 5, 6]

$$S = \{(x \mid \ \mu_C(x))\}_{x \in \mathbf{R}}$$

where $\mu_C(x)$ is called the membership function value for the element x into the fuzzy set *C.*

The fuzzy set S is called *normal* if there is at least one real point, x_0, with

$$\mu_C(x_0) = \mathit{1}$$

A fuzzy set, S, on $\mathbf{R}$ will be *convex,* if for any real numbers, *x, y,* and any t such that

$$0 \leq t \leq 1$$

we have

$$\mu_C(tx + (1-t)y) \geq min\{\mu_C(x), \ \mu_C(y)\}$$

A *fuzzy number* (*FN,* by acronym) is indeed a fuzzy set, *N,* on the real line ($\mathbf{R}$) that satisfies the above mentioned conditions of normality and convexity. I.e. a fuzzy number is merely a particular case of fuzzy set,

$$N : \mathbf{R} \to [0,1]$$

such that:

1) N is *normal,* i.e. $Height(N) = 1$.

2) N^{α} ($the\ \alpha - cut\ of\ N$) will be a closed interval, for all the values of α comprised between 0 and 1: $0 < \alpha \leq 1$. But if all $\alpha - cuts$ are closed intervals, then every fuzzy number must be a convex fuzzy set. Observe that the converse is not necessarily true.

3) The *support of N,* denoted by $supp\,(N)$, is bounded.

And it is called *positive* (*negative,* respectively), denoted as either

$$N^{-} > 0$$

or

$$N^{-} < 0$$

if its membership function μ_{N^-}, holds that $\mu_{N^-}(x) = 0$, for each $x < 0$ (or $x > 0$, respectively).

As illustrative examples of fuzzy numbers, we have: Trapezoidal Fuzzy Numbers; Triangular Fuzzy Number, as its limiting case; bell-shaped membership function (Gaussian); L-R fuzzy numbers, etc.

3. Interval Numbers

The *interval number* (*IN,* by acronym) represents an uncertain real number [5], of this type:

$$IN = [a,b] = \{x : a \leq x \leq b\}_{x \in \mathbf{R}}$$

So, a real number may be considered as a special case of IN: More concretely, the *"singleton",* also called "point interval", as

$$if\ a \in R,\ then\ IN\left(\{a\}\right) = IN([a,a])$$

We can introduce some new measures, taking a auxiliary example $N = [3/4,\ 1]$, as

- the *width* (w) *of* N, given by $w(N) = b - a$; in our case, w(N) = 1/4;
- *magnitude (m) of N,* given by m(N) = max ($|\ a\ |, |\ b\ |$) ;here, m (N) = 1;
- *image of* $N = [a,b]$ must be $im(N) = [-b,-a]$; *i.e.* $im(N) = [-4/3,\ -1]$;
- *inverse of N,* denoted by N^{-1}, as $N^{-1} = [1/b, 1/a]$; hence, $N^{-1} = [1,\ 4/3]$.

We can introduce an Hausdorff distance, between $A = [a_1, a_2]$ *and* $B = [b_1,\ b_2]$ by

$$d(A,B)=\max\{\mid a_1-b_1\mid, \mid a_2-b_2\mid\}$$

So, for ex., if $A=[2,\ 6]\,,\ and\ B=[3,\ 9]\,,$ then $d(A,B)=max\ \{1,3\}=3$
Departing from the aforementioned properties, we can define the following *operations between Interval Numbers:*

$$A*B=\{a*b: a\in A, b\in B\}$$
$$A/B=\{a/b: a\in A, b\in B,\ with\ 0\notin B\}$$
$$A+B=[a_1+b_1, a_2+b_2]$$
$$A-B=A+im(B)=[a_1,a_2]+[-b_2,-b_1]=[a_1-b_2, a_2-b_1]$$
$$A.B=[\min(a_1b_1, a_1b_2, a_2b_1, a_2b_2)\,,\ \ \max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)]$$

and imposing previously the condition that $0\notin[b_1,b_2]\,,$ also we may introduce the division between interval numbers,

$$A/B=A.B^{-1}=[a_1,a_2]\,.\,[1/b_2, 1/b_1]=$$
$$=\min(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2)\,,\ \ \max(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2)$$

We can dispose of this characterization of Fuzzy Numbers:
N will be a fuzzy number if and only if there exists a nonempty, closed interval, $[a,b]\,,$ such that:
$N(x)$ is equal to $l(x)$, if $x<a$; equal to one, if $a\leq x\leq b;$ and equal to $r(x)$, if $x>b$.
Being $r:(-\infty,a)\to[0,1]\,,$ and $l:(b,+\infty)\to[0,1]\,,$ both monotonic functions, butspecial, because they are respectively non-decreasing and non-increasing functions. And they are also continuous functions, from the right and for the left, in each case.

Its arithmetic is strongly supported on two basic principles. The fundamental will be the so-called *Extension Principle,* according to which

$$(A*B)(z)=\sup_{z=x+y}\{\min[A(x), B(y)]\}$$

And also the so-called property of *cutworthiness:*

$$A*B=\cup_{\alpha\in(0,1]}(A*B)^{\alpha}$$

Introducing now the MAX and MIN operators (representing the meet and the join respectively), we can to prove that the set of all the fuzzy numbers, provided with such operators, is a distributive lattice. Recall that a lattice consists in a poset (partially ordered set) whose non-empty finite subsets have a uniqum supremum (called join), and a uniqum infimum (called meet).

4. Complex Fuzzy Numbers

We need to introduce this new representational tool because various different reasons [5, 7, 8], which justifies their necessity and convenience, instead of a simple two element vectorial representation.

- It will be more easy for calculation.

- It is physically accurate.

- A very useful in applications, by using complex algebra.

Let U be the universe of discourse. Then, we define a *Complex Fuzzy Set* (*CFS,* by acronym), and denoted as *C,* through its membership function,

$$\mu_C(x) = r_C(x) \ \ \exp\{i\ \omega_C(x)\}$$

where i represent the imaginary unit, being r_C and ω_C both real-valued functions, with an important restriction on r_C :

$$0 \leq r_C(x) \leq 1$$

Hence, a CFS may be represented as an ordered collection of pairs of this type:

$$C = \{x \mid \mu_C(x)\}_{x \in U} = \{(x,\ \mu_C(x))\}_{x \in U}$$

Observe that sometimes certain authors use different notations for the same mathematical object; so, for instance,

$$C = \{\mu_C(x) \mid x\}_{x \in U} = \{(\mu_C(x) \mid x)\}_{x \in U}$$

Including many other confusing, as may be

$$C = \{(\mu_C(x),\ x) : x \in U\} = \{(\mu_C(x),\ x) : x \in U\} = \ ...$$

5. Interpretation of such fuzzy numbers

We can consider the precedent membership function, $\mu_C(x)$, as composed by two factors [2, 5]:

- *membership amplitude,* $r_C(x)$

and

- *membership phase,* $\omega_C(x)$

As a particular case, we may consider the case of C with a null membership function, $\mu_C(x) = 0$; for instance on the case where either $r_C(x)$, or $r_C(x)$ and $\omega_C(x) = 0$; therefore, null phase and amplitude.

We make a more detailed analysis of membership phase component, wrt the fuzzy operations, union, intersection, and so on.

Let A and B two CFS.
Then,we can define

$$\mu_{A\cup B}(x) = \ [r_A\,(x) \blacktriangleright r_B\,(x)]\ \ \exp\{i\ \omega_{_{A\cup B}}\,(x)\}$$

being $\blacktriangleright$ here some *T-conorm operator.*

And similarly (but being different) we can also define

$$\mu_{A\cap B}(x) = \ [r_A\,(x) \blacktriangleleft r_B\,(x)]\ \ \exp\{i\ \omega_{_{A\cap B}}\,(x)\}$$

being $\blacktriangleleft$ in this case some *T-norm operator.*

It remains until now without definition their respective phases,

$$\omega_{_{A\cup B}}\,(x)$$
and
$$\omega_{_{A\cap B}}\,(x)$$

To obtain such definitions, it would be convenient introduce two auxiliary functions, u and j, which permets specify the Fuzzy Union and the Fuzzy Intersection of both given fuzzy sets, A and B.

In both cases, the domain is the same fuzzy domain product, and also comparts the range:

$R_u = R_j = \{a \in \mathbf{C}:\ \ |a| = 1\}\,x\,\{b \in \mathbf{C}:\ \ |b| = 1\} \rightarrow \{d \in \mathbf{C}:\ \ |d| = 1\}$

Axioms
(which defines the u application):

At least, it must holds the following:

1) $u(a, 0) = a$

2) if $\mid b \mid\leq\mid d \mid$, then $\mid u(a, b) \mid\leq\mid u(a, d) \mid$

3) $u(a, b) = u(b, a)$

4) $u(a, u(b, d)) = u(u(a, b), d)$

The name of such axioms must be:

1) boundary conditions;

2) monotonicity;

3) commutativity;

4) associativity,

respectively.

In some cases, it will be convenient to dispose of certain additional axioms for u, as may be:

5) u is a continuous function

6) $\mid u(a,\ a) \mid>\mid a \mid$

7) If $\mid a \mid\leq\mid c \mid$and $\mid b \mid\leq\mid d \mid$, then$\mid u(a, b) \mid\leq u(c, d)$

so-called:

5) continuity;

6) supidempotency (with "p");

7) strict monotonicity, respectively.

And now we need to analyze the properties required from the phase intersection function, j.

Axioms :

At least, the j application must verifies these conditions:

1) $j(a,0)=a$

2) if $\mid b\mid\leq\mid d\mid$, then $\mid j(a,b)\mid\leq\mid j(a,d)\mid$

3) $j(a,b)=j(b,a)$

4) $j(a,j(b,d))=j(j(a,b),d)$

The adequate name of such axioms may be:

1) boundary conditions;

2) monotonicity;

3) commutativity;

4) associativity,

respectively.

In some cases, it will be convenient to dispose of certain additional axioms for j, as may be:

5) j is a continuous function

6) $\mid j(a,\ a)\mid<\mid a\mid$

7) If $\mid a\mid\leq\mid c\mid$and $\mid b\mid\leq\mid d\mid$, then$\mid u(a,b)\mid\leq u(c,d)$

so-called:

5) continuity;

6) subidempotency (with "b");

7) strict monotonicity, respectively.

6. Conclusion

Expressing by adequate definitions and operational formulae in terms of Fuzzy Numbers will be currently a very inspired and useful idea. In fact, their applications extends to many different and promising fields, as may be on Logic, Mathematical Analysis, Physics, Quantum Theory, Theoretical Computer Science, Automata Theory, and so on.

Acknowledgements

In this, my published paper number 207 until now, I wish to dedicate my work to many friends and scientific researchers, as for instance Neculai Andrei, Barna Lászlo Iántovics, Hari Srivastava, Shigeyoshi Owa, Afrodita Iorgulescu, Shu-Kun Lin, Joel Ratsaby, Eugen Mandrescu, Cabiria Andreian Cazacu, Mircea Cazacu, Joe Rosen, Viorel Barbu, Solomon Marcus, Marcelina

Mocanu, Matthias Dehmer, Florin Nichita, Themistocles Rassias, Adrian Constantinescu, Dorin Andrica, Petru Mocanu, Bogdan Patrut, Gheorge Paun, Michel Petitjean, Bazil Parv, Mirela Stefanescu, Paul Dan Cristea, Janos Fodor, and many others. And also to my family; without its support, this could not have been possible.

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