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Fuzzy Mean-Variance portfolio selection problems

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Abstract

In this paper the portfolio selection problem in a fuzzy environment is studied, in particular situations in which the returns of the assets are modelled by specific types of fuzzy numbers. Extensions of the classical Markowitz's portfolio selection model are proposed, using fuzzy measures of the profit and risk of the studied portfolios. Numerical examples are given to illustrate these models.
Key words: Fuzzy number, Mean-Variance model, Portfolio selection

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1. Introduction

Markowitz's mean-variance model has been one of the principal methods of financial theory and assets selection. It allows us to represent the investor's problem as a mathematical programming problem. In the case of the ratios of the returns were normally distributed, the problem can be represented as a quadratic programming problem. On the other hand, fuzzy theory allows us to represent the investor's preferences, in particular it can be used in the

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portfolio selection problem. Many authors have integrated these techniques and have proposed portfolio selection problems in fuzzy environments.

This paper is organized as follows: in section 2 the principal and necessary concepts about fuzzy numbers that will be used, are explained. In section 3 principal portfolio selection problems are described. In section 4 these models are particularized in some specific cases and finally in section 5 numerical examples are provided as an illustration of the previously studied models.

2. Some concepts about fuzzy numbers

A fuzzy number A is a fuzzy set of the real line $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. The α-level set of a fuzzy number A is defined by $[A]^\alpha = \{x \in \mathbb{R} | A(x) \geq \alpha\}$ if $0 < \alpha \leq 1$ and $[A]^\alpha = cl\{x \in \mathbb{R} | A(x) > 0\}$ if $\alpha = 0$, where cl denotes the closure of the support of A. If A is a fuzzy number then its α-level set is a convex and compact interval of the real line.

There are different ways to define the expected value and variance of a fuzzy number, for example, using the concepts of possibility, necessity and credibility (see Liu, 2004) or using the concept of α-level set (see Carlsson and Fullér, 2001). We will use this second way in our development.

Definition 1. *Let A be a fuzzy number with α-level set $[a_1(\alpha), a_2(\alpha)]$. The crisp possibilitistic mean value of A is defined as:*

$$E(A) = \int_0^1 \alpha(a_1(\alpha) + a_2(\alpha))d\alpha \tag{1}$$

It is easy to prove that given two fuzzy numbers A and B and a real number λ, the following properties are verified:

1. $E(A+B) = E(A) + E(B)$
2. $E(\lambda A) = \lambda E(A)$

Definition 2. *The possibilistic variance of a fuzzy number A with α-level set $[a_1(\alpha), a_2(\alpha)]$ is defined as:*

$$Var(A) = \frac{1}{2}\int_0^1 \alpha(a_2(\alpha) - a_1(\alpha))^2 d\alpha \tag{2}$$

The variance of A is defined as the expected value of the squared deviations between the arithmetic mean and the endpoints of its levels sets.

The standard deviation of A is $\sigma(A) = \sqrt{Var(A)}$.

Definition 3. *Given two fuzzy numbers A and B with α-level sets $[a_1(\alpha), a_2(\alpha)]$ and $[b_1(\alpha), b_2(\alpha)]$ respectively, we define the covariance between A and B, and we denote $Cov(A, B)$, as:*

$$Cov(A, B) = \frac{1}{2}\int_0^1 \alpha(a_2(\alpha) - a_1(\alpha))(b_2(\alpha) - b_1(\alpha))d\alpha \tag{3}$$

The following result shows how we can calculate the variance of a linear combination of fuzzy numbers:

Theorem 4. *Let λ and μ two real numbers and A and B two fuzzy numbers, then $V(\lambda A+\mu B) = \lambda^2 V(A)+\mu^2 V(B)+2|\lambda||\mu|Cov(A, B)$, where the addition and multiplication by a scalar of fuzzy numbers is defined by the sum-min extension principle.*

For the proof see Carlsson and Fullér, 2001.

3. Fuzzy Mean-Variance models

The mean-variance model of portfolio selection with uncertain returns was introduced by Liu as an extension of the Markowitz's classical mean-variance model in uncertain environment. Markowitz's model provides a parametric optimization model that re-presents different and significant practical situations and that shows the investor's conflict: obtain a high profit or a low risk? Normally the profit of the investor's portfolio is represented as the expected value of the random variable which represents the total amount of the investment, another question is how we can measure the risk. There are many developments about this issue, the first approximation is using the variance as a measure of risk, this is the way we are going to follow in this paper, other measures can be the semi-variance, asymmetric risk measures, a quantil or a functional expressions of the previous one, etc.

Let us suppose that we want to build a portfolio using n assets. If we denote by A_i, $i = 1, ..., n$, the return rate of asset i and w_i the proportion of total amount of funds invested in the ith asset, with $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, the total return of our portfolio will be $\epsilon = \sum_{i=1}^{n} w_i A_i$. Usually $w_i \geq 0$ but we can also have restrictions about the maximum (M_i) and minimum (m_i) proportion of total amount that will be destined to the ith asset, i.e. $m_i \leq w_i \leq M_i$. Typical targets that are presented in the optimal portfolio selection can be maximization of the expected value of the portfolio, the maximization of the benefit of the portfolio (if we consider transaction costs), minimization of the variance or other risk measure, etc.

Many researches have studied the optimal portfolio problem taking into account the fuzziness aspect of uncertainty and the human decisions. Principal contributions in this sense are due to [Liu and Iwamura, 1998] who study chance constrained programming with fuzzy environment using the possibility as a risk measure, [Peng, Mok and Tse, 2005], who study portfolio selection problems using the concepts of expected value of a fuzzy variable (defined as a difference of integrals of the credibility measure) and fuzzy variance, [Chen, Chen, Fang, and Wang, 2006] who study a possibilistic mean VaR model for portfolio selection, in the same line as the previous ones we can find Huang's works (2007) about the portfolio selection in fuzzy environment using fuzzy retuns, [Xu and Zhai, 2009] who study the optimal portfolio selection using fuzzy return rates and some indexes as the measurement of the portfolio variability, etc.

4. Particular cases of rates of return

In this section we consider especial cases of the previous problems using specifical fuzzy numbers. For this purpose, we have to compute the mean, variance and covariance of the fuzzy numbers under consideration.

4.1. Trapezoidal fuzzy numbers

Definition 5. *A is said to be a trapeziodal fuzzy number if its membership function is:*

$$\mu(x) = \begin{cases} \frac{x-r_1}{r_2-r_1}, & if \quad r_1 \leq x \leq r_2 \\ 1, & if \quad r_2 \leq x \leq r_3 \\ \frac{x-r_4}{r_3-r_4}, & if \quad r_3 \leq x \leq r_4 \\ 0, & otherwise \end{cases}$$

with $r_1 \leq r_2 \leq r_3 \leq r_4$. Its parametrization can be represented by $A = (r_1, r_2, r_3, r_4)$.

The α-level set of A is $[A]^\alpha = [r_1 + \alpha(r_2 - r_1), r_4 - \alpha(r_4 - r_3)]$. Using this information we calculate the mean and the variance as we defined in (1) and (2): $E(A) = \frac{1}{6}(r_1 + r_4) + \frac{1}{3}(r_2 + r_3)$ is the expected value and the variance is: $V(A) = \frac{(r_1-r_4)^2}{4} + \frac{(r_3-r_4+r_1-r_2)^2}{8} + \frac{(r_4-r_3)(r_3-r_4+r_1-r_2)}{3}$

Let $A_k = (r_{1k}, r_{2k}, r_{3k}, r_{4k})$ (with $r_{1k} \leq r_{2k} \leq r_{3k} \leq r_{4k}$), $k = i, j$ be two trapezoidal fuzzy numbers, the covariance between A_i and A_j using definition (3) is: $Cov(A_i, A_j) = \frac{T_{1i}T_{1j}}{4} - \frac{T_{1i}T_{2j}+T_{2i}T_{1j}}{6} + \frac{T_{2i}T_{2j}}{8}$. with $T_{1k} = r_{4k} - r_{1k}$, $T_{2k} = r_{4k} - r_{3k} + r_{2k} - r_{1k}$, $k = i, j$.

Let us consider n trapezoidal fuzzy numbers representing the rates of return of n assets: $A_i = (r_{1i}, r_{2i}, r_{3i}, r_{4i})$, with $r_{1i} \leq r_{2i} \leq r_{3i} \leq r_{4i}$ and we build the portfolio for $i = 1, ..., n$. Using the properties of the mean and the variance we have seen in section two we can calculate the mean and the variance of the portfolio: $E(\epsilon) = \sum_{i=1}^n w_i E(A_i) = \sum_{i=1}^n w_i \left(\frac{r_{1i}+r_{4i}}{6} + \frac{r_{2i}+r_{3i}}{3}\right)$ and $V(\epsilon) = \sum_{i=1}^n w_i^2 V(A_i) + 2\sum_{i<j} |w_i w_j| Cov(A_i, A_j) = \sum_{i=1}^n w_i^2 \left(\frac{T_{1i}^2}{4} - \frac{T_{1i}T_{2i}}{3} + \frac{T_{2i}^2}{8}\right) + 2\sum_{i<j} w_i w_j \left(\frac{T_{1i}T_{1j}}{4} - \frac{T_{1i}T_{2j}+T_{2i}T_{1j}}{6} + \frac{T_{2i}T_{2j}}{8}\right)$ respectively.

4.2. Triangular fuzzy numbers

Let us consider triangular fuzzy numbers. This is a particular case of trapezoidal fuzzy number.

Definition 6. *A is said to be a triangular fuzzy number if its membership function is:*

$$\mu(x) = \begin{cases} \frac{x-r_1}{r_2-r_1}, & if \quad r_1 \leq x \leq r_2 \\ \frac{x-r_3}{r_2-r_3}, & if \quad r_2 \leq x \leq r_3 \\ 0, & otherwise \end{cases}$$

with $r_1 \leq r_2 \leq r_3$. Its parametrization can be represented by $A = (r_1, r_2, r_3)$.

The α-level set of A is $[A]^\alpha = [r_1 + \alpha(r_2 - r_1), r_3 - \alpha(r_3 - r_2)]$. Using this information we calculate the mean and the variance as we have define above: $E(A) = \frac{r_1+4r_2+r_3}{6}$ and $V(A) = \frac{(r_3-r_1)^2}{24}$

Given two triangular fuzzy numbers $A_k = (r_{1k}, r_{2k}, r_{3k})$ with $r_{1k} \leq r_{2k} \leq r_{3k}$, $k = i, j$ their covariance is: $Cov(A_i, A_j) = \frac{1}{24} T_i T_j$, where $T_k = r_{3k} - r_{1k}$, $k = i, j$.

Let us consider n triangular fuzzy numbers representing the return rates of n assets: $A_i = (r_{1i}, r_{2i}, r_{3i})$, with $r_{1i} \leq r_{2i} \leq r_{3i}$, for all $i = 1, ..., n$, and we build the portfolio. The mean and the variance of the portfolio are:

$$E(\epsilon) = \sum_{i=1}^{n} w_i E(A_i) = \frac{1}{6} \sum_{i=1}^{n} w_i \left(r_{1i} + 4r_{2i} + r_{3i}\right)$$

$$V(\epsilon) = \frac{1}{24} \sum_{i=1}^{n} w_i^2 (r_{3i} - r_{1i})^2 + \frac{1}{12} \sum_{i<j} w_i w_j (r_{3i} - r_{1i})(r_{3j} - r_{1j})$$

4.3. L-R type fuzzy numbers

Definition 7. *A is said to be a L-R fuzzy number if its membership function is:*

$$\mu(x) = \begin{cases} L(\frac{q_- - x}{\lambda}), & if \quad q_- - \lambda \leq x \leq q_- \\ 1, & if \quad q_- \leq x \leq q_+ \\ R(\frac{x - q_+}{\mu}), & if \quad q_+ \leq x \leq q_+ + \mu \\ 0, & otherwise \end{cases}$$

where L, R: $[0,1] \to [0,1]$*, with* $L(0) = R(0) = 1$ *and* $L(1) = Q(1) = 0$*,* $[q_-, q_+]$ *is the peak of A with* $q_- \leq q_+$ *and* $\lambda, \mu > 0$*. Its parametrization is:* $A = (q_+, q_-, \lambda, \mu)_{LR}$.

If R and L are strictly decreasing the α-level set is $[q_- - \lambda L^{-1}(\alpha), q_+ + \mu R^{-1}(\alpha)]$, with $\alpha \in [0,1]$. The mean and variance of A are:

$$E(A)=\frac{q_- - \lambda\int_0^1 \alpha L^{-1}(\alpha)d\alpha + q_+ + \mu\int_0^1 \alpha R^{-1}(\alpha)d\alpha}{2}$$

$$V(A)=\frac{(q_+ - q_-)^2}{4}+\frac{1}{2}\int_0^1 \alpha[\mu^2(R^{-1}(\alpha))^2+\lambda^2(L^{-1}(\alpha))^2-2\lambda\mu R^{-1}(\alpha)L^{-1}(\alpha)]d\alpha+$$
$$+(q_+ - q_-)\int_0^1 \alpha(\mu R^{-1}(\alpha)-\lambda L^{-1}(\alpha))d\alpha$$

Given two L-R fuzzy numbers $A_k=(q_{k+},q_{k-},\lambda_k,\mu_k)_{LR}$, $k=i,j$, their covariance is:

$$Cov(A_i,A_j)=\frac{T_iT_j}{4}+\frac{T_i}{2}\int_0^1 \alpha H_j(\alpha)d\alpha+\frac{T_j}{2}\int_0^1 \alpha H_i(\alpha)d\alpha+\frac{1}{2}\int_0^1 \alpha H_i(\alpha)H_j(\alpha)d\alpha$$

where $T_k=q_{k+}-q_{k-}$ and $H_k(\alpha)=\mu_k R_k^{-1}(\alpha)+\lambda_k L_k^{-1}(\alpha)$, for $k=i,j$.

4.4. Generalized bell shaped fuzzy numbers

Definition 8. *A is said to be a generalized bell shaped fuzzy number if its membership function is: $\mu(x)=\frac{1}{1+|\frac{x-c}{a}|^{2b}}, x\in\mathbb{R}$, where the parameter a represents the width of the curve, b is usually positive and c locates the center of the curve.*

Its α-level set is: $\left[|a|\sqrt[2b]{\frac{1-\alpha}{\alpha}}-c, |a|\sqrt[2b]{\frac{1-\alpha}{\alpha}}+c\right]$ and for $b>1/4$, the mean and the variance are: $E(A)=|a|\Gamma\left(2-\frac{1}{2b}\right)\Gamma\left(1+\frac{1}{2b}\right)$ and $V(A)=c^2$ respectively.

Given two generalized bell shaped fuzzy numbers $A_k=(a_k,b_k,c_k)$, $k=i,j$ their covariance is: $Cov(A_i,A_j)=c_ic_j$.

Let us consider n generalized bell-shaped fuzzy numbers representing the return rates of n assets: $A_i=(a_i,b_i,c_i)$, $b_i>1/4$, $i=1,...,n$, and we build the portfolio. The mean and the variance of the portfolio ϵ are:

$$E(\epsilon)=\sum_{i=1}^{n} w_i|a_i|\Gamma\left(2-\frac{1}{2b_i}\right)\Gamma\left(1+\frac{1}{2b_i}\right)$$

$$V(\epsilon) = \sum_{i=1}^{n} w_i^2 c_i^2 + 2 \sum_{i<j} w_i w_j c_i c_j$$

5. Some fuzzy portfolio selection models and numerical examples

In this section we provide numerical examples of some optimization fuzzy models like as described in section 3 using the fuzzy numbers we have spoken about previously.

Let us consider the following four trapezoidal fuzzy numbers that represent the returns of four different assets: $A_1 = (0.03, 0.04, 0.07, 0.08)$, $A_2 = (0.03, 0.07, 0.075, 0.08)$, $A_3 = (0.048, 0.068, 0.07, 0.08)$ and $A_4 = (0.04, 0.05, 0.06, 0.07)$. Our target is know the proportion of the total amount that will be destined to each one. Let $c = (0, 0.001, 0.001, 0.002)$ be the vector of the transaction costs. We obtain the mean and the variance of the portfolio:

$$E(\epsilon) = 0.055w_1 + 0.0667w_2 + 0.0673w_3 + 0.055w_4$$

$V(\epsilon) = 3.4167 \cdot 10^{-4} w_1^2 + 1.2812 \cdot 10^{-4} w_2^2 + 4.85 \cdot 10^{-5} w_3^2 + 7.5 \cdot 10^{-5} w_4^2 + 3.9166 \cdot 10^{-4} w_1 w_2 + 2.3666 \cdot 10^{-4} w_1 w_3 + 3.1666 \cdot 10^{-4} w_1 w_4 + 1.5750 \cdot 10^{-4} w_2 w_3 + 1.9167 \cdot 10^{-4} w_2 w_4 + 1.1667 \cdot 10^{-4} w_3 w_4$

And now we consider the problems:

$$(P1) \begin{cases} max \quad E(\epsilon) \\ s.t. \quad V(\epsilon) \leq 0.00005 \\ \qquad \sum_{i=1}^{n} w_i = 1 \\ \qquad w_i \geq 0, i = 1, ..., n \end{cases} , \quad (P2) \begin{cases} min \quad V(\epsilon) \\ s.t. \quad E(\epsilon) \geq 0.05 \\ \qquad \sum_{i=1}^{n} w_i = 1 \\ \qquad w_i \geq 0, i = 1, ..., n \end{cases} .$$

The solution of (P1) is: $(w_1, w_2, w_3, w_4) = (0, 0, 1, 0)$, objective function=0.0673 and the solution of (P2): $(w_1, w_2, w_3, w_4) = (0, 0.8499, 0.11, 0.04)$, objective function=$1.1503 \cdot 10^{-4}$.

$$(P3)\begin{cases} max \quad E(\epsilon) - \sum_{i=1}^{n} c_j A_j \\ s.t. V(\epsilon) \leq 0.05 \\ \qquad \sum_{i=1}^{n} w_i = 1 \\ \qquad w_i \geq 0, i = 1, ..., n \end{cases}$$

Its solution is: $(w_1, w_2, w_3, w_4) = (0, 0, 1, 0)$, objective function=0.0663.

Let us consider now a portfolio consisting of four assets whose returns are modeled by the following triangular fuzzy numbers: $A_1 = (0.03, 0.04, 0.05)$, $A_2 = (0.03, 0.07, 0.08)$, $A_3 = (0.04, 0.06, 0.08)$ and $A_4 = (0.04, 0.05, 0.07)$. The mean and the variance of this portfolio are:

$$E(\epsilon) = \frac{1}{6}(0.24w_1 + 0.39w_2 + 0.36w_3 + 0.31w_4)$$

and

$V(\epsilon) = 10^{-4}w_1^2 + 6.25 \cdot 10^{-4}w_2^2 + 4 \cdot 10^{-4}w_3^2 + 2.25 \cdot 10^{-4}w_4^2 + \frac{1}{12}(10^{-3}w_1w_2 + 8 \cdot 10^{-4}w_1w_3 + 6 \cdot 10^{-4}w_1w_4 + 0.002w_2w_3 + 0.0015w_2w_4 + 0.0012w_3w_4)$

Now we consider the problems:

$$(P1)\begin{cases} max \quad E(\epsilon) \\ s.t. \quad V(\epsilon) \leq 0.005 \\ \qquad \sum_{i=1}^{n} w_i = 1 \\ \qquad w_i \geq 0, i = 1, ..., n \end{cases}, \quad (P2)\begin{cases} min \quad V(\epsilon) \\ s.t. \quad E(\epsilon) \geq 0.05 \\ \qquad \sum_{i=1}^{n} w_i = 1 \\ \qquad w_i \geq 0, i = 1, ..., n \end{cases}.$$

The solution of (P1) is: $(w_1, w_2, w_3, w_4) = (0, 1, 0, 0)$, objective function=0.0650 and the solution of (P2): $(w_1, w_2, w_3, w_4) = (0.1002, 0.1669, 0.2599, 0.47)$, objective function=$1.384 \cdot 10^{-4}$.

Finally, we will consider a portfolio of generalized bell-shaped fuzzy numbers: $A_1 = (3, 6, 8)$, $A_2 = (2, 3, 4)$ and $A_3 = (1, 2, 3)$. The mean and the variance of the portfolio ϵ are:

$$E(\epsilon) = 2.7817w_1 + 1.7453w_2 + 0.8330w_3$$

and $V(\epsilon) = 64w_1^2 + 16w_2^2 + 9w_3^2 + 32w_1w_2 + 24w_1w_3 + 12w_2w_3$.

Now we consider the problems:

$$(P1)\begin{cases} max \quad E(\epsilon) \\ s.t. \quad V(\epsilon) \leq 70 \\ \qquad \sum_{i=1}^{n} w_i = 1 \\ \qquad w_i \geq 0, i = 1, ..., n \end{cases} \quad , \quad (P2)\begin{cases} min \quad V(\epsilon) \\ s.t. \quad E(\epsilon) \geq 2 \\ \qquad \sum_{i=1}^{n} w_i = 1 \\ \qquad w_i \geq 0, i = 1, ..., n \end{cases} .$$

The solution of (P1) is: $(w_1, w_2, w_3) = (1, 0, 0)$, objective function=2.7817 and the solution of (P2) is: $(w_1, w_2, w_3) = (0.2458, 0.7542, 0)$, objective function=18.8990.

6. Conclusion

In this paper we have consider several portfolio selection problems using Markowitz's methodology with the particularity that the measures of the benefit and risk of the portfolio are represented by fuzzy measures. Proposed models can be solved using quadratic and non-linear programming. Numerical examples of the considered models are showed as an illustration.

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