A different view on using measure theoretical approach for a class of optimal control problems

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Abstract

In this article we try to make some changes on the procedure of using measure theoretical approach for overcoming some problems that may occur when using this approach to solve a class of optimal control problems. Since the measure theoretical approach based on the deformation problem to a linear programming, our changes decrease the number of variables of deformed problem. Numerical examples show the effect of changes on the performance of the approach.

Keywords: optimal control, measure theory, linear programming, approximation
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1 Introduction

L. C. Young [27] unique ideas for finding generalized solution in the calculus of variations formed the starting point for using some concepts of measure theory for solving a wide range of problems in applied mathematics. Specially this idea persuaded Rubio to extend it to solve classical nonlinear optimal control problems [26]. Because of its flexibility, this method has been extended and improved by many researches for solving a variety of problems; Miscellaneous problems in optimal control area [6, 7, 8, 20, 17], optimal control governed by distributed parameter systems [2, 15, 18, 19, 16], optimal shape designing problems [13, 14, 11, 12, 21, 22, 23, 24], optimal path planning problems [3] and even solving some problems in numerical

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computations [1, 4, 5, 9, 10].
The most important problem in implementing of the approach is related to the number of state and control variables. The approach based on the linear treatment of nonlinear systems where by a metamorphosis in optimal control problem near optimal solution can be obtained from a linear programming problem and the technology and cost coefficients of this linear programming can be extracted from partitioning of the sets, where the control and state functions take their values in them. For clarify the discussed problem and proposing a appropriate approach we concentrate on a linear control system as

$$\sum_{i=0}^{n} a_i(t)x^{(i)}(t) = f(t,u), \tag{1}$$

where $f(\cdot,\cdot)$ is an arbitrary nonlinear function, with initial and final conditions as

$$x^{(i)}(t_0) = x_{i+1}^{\circ}, \quad x^{(i)}(t_f) = x_{i+1}^{f}, \quad i = 0, 1, \cdots, n-1, \tag{2}$$

where $a_i(\cdot)$, $i = 0, 1, \cdots, n$ are infinitely differentiable functions on time interval $\mathcal{T} = [t_0, t_f]$ and
(i) $u(\cdot)$ is the control function which is a measurable function and takes its values within the set $\mathcal{U}$, a compact subset of $\mathbb{R}$.
(ii) $x(\cdot)$ is the state function which is n times differentiable function and takes its values within the compact set $\mathcal{X} \subset \mathbb{R}$.

We call the pair $p = (x(\cdot), u(\cdot))$, admissible pair if its components satisfy in the above conditions (i) and (ii) and the equation (1). Let $\mathcal{P}$ be the set of all admissible pairs and this set is not empty. For applying a measure theoretical approach to obtain approximate optimal trajectory and control with cost function

$$\int_{t_0}^{t_f} f_\circ(t, x(t), u(t))dt, \tag{3}$$

governed by (1)-(2), where $f_\circ$ is continuous function on $\Omega = \mathcal{T} \times \mathcal{X} \times \mathcal{U}$, we need to convert the linear differential equation (1) to a first order linear system, thus by defining the following functions

$$x_1(t) := x(t),$$
$$x_i(t) := x'_{i-1}(t), \quad i = 2, 3, \cdots, n,$$

we substitute the n order linear system (1) by the following first order linear system with n equations and $n+1$ unknown functions as

$$\begin{aligned} x_i'(t) &= x_{i+1}(t), \quad i = 1, 2, \cdots, n-1, \\ x_n'(t) &= f(t,u) - a_0(t)x_1(t) - \cdots - a_{n-1}(t)x_n(t), \end{aligned} \tag{4}$$

with boundary conditions

$$x_i(t_0) = x_i^\circ, \quad x_i(t_f) = x_i^f, \quad i = 1, \cdots, n, \tag{5}$$

and if we suppose that each state function $x_i(t)$ $i = 1, \cdots, n$, takes its values in the bounded set $\mathcal{X}_i$ $i = 1, \cdots, n$, and the control function takes its values in the bounded set $\mathcal{U}$ then the procedure of approach necessitate us to choose ν_i, $i = 1, 2, \cdots, n$ and γ nodes from the sets $\mathcal{X}_i$ $i = 1, \cdots, n$, and $\mathcal{U}$, respectively, and finally, we must solve a linear programming with $\tau\gamma\Pi_{i=1}^n \nu_i$ variables, where τ is the number of nodes in the time interval. We tend to make some changes in applying measure theory for decreasing the number of variables in the linear programming.

2 Metamorphosis

We consider an equivalent weak form of equation (1) by multiplying this equation by the test function ϕ as

$$\sum_{j=0}^{n} \phi(t)a_j(t)x^{(j)}(t) = \phi(t)f(t,u), \ \forall \phi \in C^n(\mathcal{T}),$$

and by integrating the bilateral of the above equation on the interval $\mathcal{T}$

$$\sum_{j=0}^{n} \int_{t_0}^{t_f} \varphi_j(t)x^{(j)}(t)dt = \int_{t_0}^{t_f} \phi(t)f(t,u)dt,$$

where $\varphi_j(t) := \phi(t)a_j(t)$. Now integration by parts on the above equality concludes that

$$\sum_{j=0}^{n}\sum_{i=0}^{j-1}(-1)^i\varphi_j^{(i)}(t)x^{(j-i-1)}(t)|_{t_0}^{t_f} + \int_{t_0}^{t_f}(\sum_{j=0}^{n}(-1)^j\varphi_j^{(j)}(t)x(t))dt = \int_{t_0}^{t_f} \phi(t)f(t,u)dt,$$

and by rearrangment of the terms in the latest equality we obtain

$$\int_{t_0}^{t_f} G_\phi(t,x,u)dt = \alpha_\phi, \ \forall \phi \in C^n(\mathcal{T}), \tag{6}$$

where

$$G_\phi(t,x,u) = (\sum_{j=0}^{n}(-1)^{j+1}\varphi_j^{(j)}(t)x(t)) + \phi(t)f(t,u),$$

and

$$\alpha_\phi = \sum_{j=0}^{n}\sum_{i=0}^{j-1}(-1)^i(\varphi_j^{(i)}(t_f)x^{(j-i-1)}(t_f) - \varphi_j^{(i)}(t_0)x^{(j-i-1)}(t_0)).$$

For each admissible pair p, we define the linear functional

$$\Lambda_p : F \to \int_{t_0}^{t_f} F(t,x(t),u(t))dt, \quad F \in C(\Omega), \tag{7}$$

where this functional has some useful properties: It is well-defined, linear, positive, and easily it can be shown that it is uniformly continuous.

Proposition 1 *The transformation $p \to \Lambda_p$ of an admissible pair p in P into the linear mappings Λ_P defined by (7) is an injection.*

Proof: See [26]. □

Whereas based on Riesz's representation theorem, there exists a unique positive Radon measure μ on Ω such that

$$\Lambda_p(F) = \int_\Omega F d\mu, \qquad \forall F \in C(\Omega), \tag{8}$$

therefore, minimizing functional (3) over $\mathcal{P}$ is equivalent to the minimization of

$$I : \mu \to \mu(f_\circ),$$

$$I(\mu) = \int_\Omega f_\circ d\mu \equiv \mu(f_\circ), \tag{9}$$

over the set of measures μ corresponding to the admissible pairs p, which satisfies

$$\mu(G_\phi) = \alpha_\phi. \tag{10}$$

and we call this set as Q. Since all the functions in (10) are linear with respect to the measure μ, this minimization problem is an infinite-dimensional linear programming problem, where the required measure μ is positive. We call the set of all positive Radon measures on Ω as $\mathcal{M}^+(\Omega)$, and it can be shown that when the space $\mathcal{M}^+(\Omega)$ be topologized by the *weak**-topology, Q is compact. Thus, the functional $I : Q \to \mathbb{R}$ defined by (9) is a linear continuous functional on the compact set Q, and takes its minimum value on Q. It means that the measure-theoretical problem, which seeks minimum of the functional (9) over the subset $\mathcal{M}^{+}(\Omega)$, possesses a solution μ^* in Q and, therefore, to find solution of the infinite-dimensional linear programming problem (9)-(10) one can take an alternative way and construct a piecewise constant function $u(\cdot)$ corresponds to the optimal measure μ^*.

But, firstly it is required to show that approximation of the solution of the problem (9)-(10) by a solution from a finite-dimensional linear program of sufficiently large dimension is possible. For this purpose, we begin with considering the minimization problem (9) not over the whole compact set Q yet over a subset of it, requiring that only a finite number of constraints in (10) be satisfied and suppose that the set of test functions $\{\phi_m,\ m \in \mathbb{N}\}$ is as a basis of space $C^n(\mathcal{T})$.

Proposition 2 *Let $Q(M)$ be a subset of $\mathcal{M}^{+}(\Omega)$ consists of all measures satisfying (10), and*

$$\mu(G_{\phi_m}) = \alpha_{\phi_m}, \qquad m = 1, 2, \cdots, M. \tag{11}$$

If $\eta(M) = \inf_{Q(M)} \mu(f_\circ)$, and $\eta = \inf_Q \mu(f_\circ)$, then $\eta(M) \to \eta$ as $M \to \infty$.

Proof: The proof can be readily obtained from the proof of Proposition 2 in [3] by a simple restriction.□

From the Theorem A.5 in [26], one can characterize a measure, say μ^*, in the set $Q(M)$ at which the function $\mu \to \mu(f_\circ)$ takes its minimum; It follows from a result in [25] that says:

Proposition 3 *The measure μ^* in the set $Q(M)$ at which the function $\mu \to \mu(f_\circ)$ attains its minimum has the form*

$$\mu^* = \sum_{k=1}^{M} \varrho_k^* \delta(z_k^*), \tag{12}$$

with $z_k^* \in \Omega$, and the coefficients $\varrho_k^* \geq 0, k = 1, 2, \cdots, M$.
Note that $\delta(z)$ is a unitary atomic measure, characterized by $\delta(z)(H) = H(z)$, where $H \in C^1(\Omega)$ and $z \in \Omega$.

So far, the measure theoretical optimization problem has been transformed into an equivalent nonlinear optimization problem, in which the unknowns are coefficients ϱ_k^* and supports $\{z_k^*\},\ k = 1, 2, \cdots, M$. Solving this problem would become more convenient if we could minimize function $\mu \to \mu(f_\circ)$ only with respect to the coefficients

$\varrho_k^*,\ k=1,2,\cdots,M$ in (12), which is a linear programming problem. Therefore, as the support of the optimal measure is not known yet, if we approximate this support by introducing a dense set in Ω, it would be a right step.

Proposition 4 *Let ω be a countable dense subset of Ω. Given $\varepsilon>0$, a measure $\lambda \in \mathcal{M}^+(\Omega)$ can be found such that*

$$|(\mu^*-\lambda)(f_\circ)|<\varepsilon,$$

and

$$|(\mu^*-\lambda)(G_{\phi_m})|<\varepsilon,\ m=1,2,\cdots,M.$$

The measure λ has the form

$$\lambda=\sum_{k=1}^{M}\varrho_k^*\delta(z_k),$$

where the coefficients ϱ_k^* are the same as those of the optimal measure (12) and $z_k \in \omega$.

Proof: Trivially, for each $m=1,2,\cdots,M$ it can be deduced that,

$$|(\mu^*-\lambda)G_{\phi_m}|=|\sum_{k=1}^{M}\varrho_k^*[G_{\phi_m}(z_k^*)-G_{\phi_m}(z_k)]|$$

$$\leq \max_{m,k}|G_{\phi_m}(z_k^*)-G_{\phi_m}(z_k)|\sum_{k=1}^{M}\varrho_k^*,$$

and for $f(z)=1$:

$$\mu^*(f)=\sum_{k=1}^{M}\varrho_k^*\delta(z_k^*)(f)=\sum_{k=1}^{M}\varrho_k^*.$$

Now since

$$\mu^*(f)=\int_\Omega f d\mu^*=\int_{t_0}^{t_f}dt=(t_f-t_0),$$

thus

$$|(\mu^*-\lambda)G_{\phi_m}|\leq (t_f-t_0)\max_{m,k}|G_{\phi_m}(z_k^*)-G_{\phi_m}(z_k)|.$$

Since $G_{\phi_m}\in C^1(\Omega),\ m=1,2,\cdots,M$ and z_k's are in ω, which is a dense set in Ω, by choosing z_k's sufficiently near to z_k^*'s such that

$$\max_{m,k}|G_{\phi_m}(z_k^*)-G_{\phi_m}(z_k)|<\frac{\varepsilon}{(t_f-t_0)},$$

the second inequality in the proposition will be achieved. In a similar manner, the other inequalities can be deduced as well and the proof would be complete.□

Yet, we showed that the infinite-dimensional linear programming (9) with restrictions (10) can be approximated by a finite-dimensional linear programming provided that $z_i, i = 1, 2, \cdots, M$ belong to ω. For constructing set ω as a dense subset of Ω, one can cover the set Ω with a grid defined by taking all points in Ω as $z_j = (t_j, x_j, u_j)$ and number them sequentially from 1 to M.

Following this manner and using the results derived from propositions 3-4, we re-formulate the problem (9)-(10) as linear programming

$$\text{Minimize} \quad \sum_{k=1}^{M} \varrho_k f_\circ(z_k), \tag{13}$$

over the set $\varrho_k \geq 0, k = 1, 2, \cdots, M$ subject to:

$$\sum_{k=1}^{M} \varrho_k G_{\phi_m}(z_k) = \alpha_{\phi_m}, \qquad m = 1, 2, \cdots, M. \tag{14}$$

The procedure of constructing piecewise constant control functions from the solution of linear programming problem (13)-(14) which approximate the action of the optimal measure (12) is based on the analysis given in Chap.5 of [26]. Through this method, it is only required to construct approximate control vector function $u(\cdot)$, since the trajectory vector function can then be simply approximated numerically corresponding to the solution of initial value problem (1)-(2). Subsequently, test functions in the constraints (14) can be chosen following the approach described in [3]. Now, we have to solve a linear programming problem with $M = \tau\gamma\nu$ numbers of variables where τ, γ and ν are the number of nodes in time interval, $\mathcal{X}$ and $\mathcal{U}$, respectively.

3 Numerical results

In this section by some examples we show the efficiency of performed changes.

Example 1. Consider the linear optimal control problem which is minimizing the functional

$$\frac{1}{2}\int_0^1 u^2(t)dt,$$

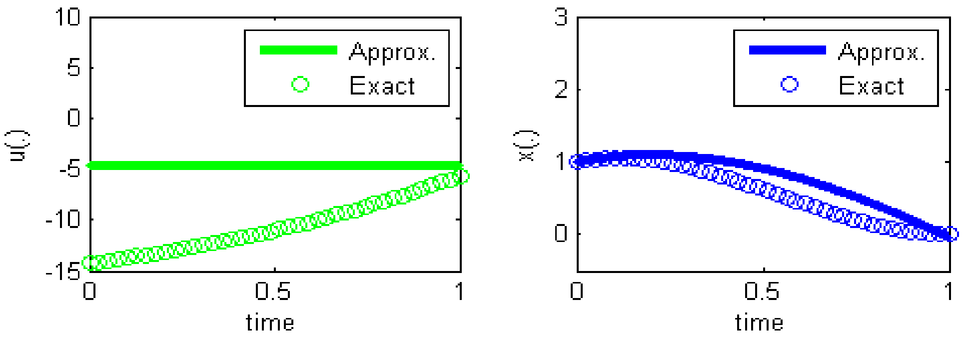

Figure 1: Exact and approximate optimal control and state functions in Example 1.

governed by the second order linear differential equation

$$x'' + x = u, \tag{15}$$

and boundary conditions

$$x(0) = x'(0) = 1, \quad x(1) = x'(1) = 0.$$

Usual implementation of measure theoretical approach needs to convert the above linear differential equation to the first order linear system

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -x_2(t) + u(t), \end{aligned}$$

and the boundary conditions

$$x_1(0) = x_2(0) = 1, \; x_1(1) = x_2(1) = 0.$$

Choosing 20 nodes from the subsets of $\mathbb{R}$, that the time variable, state functions $x_1(\cdot)$ and $x_2(\cdot)$ and control function $u(\cdot)$ take their values in them, give rise to a linear programming with 16×10^4 variables. By multiplying test functions $\phi(\cdot)$ in (15) and integration by parts we obtain an integral form equation as

$$\int_0^1 (\phi''(t) - \phi(t))x(t) + \phi(t)u(t))dt = \phi(0) - \phi'(0), \quad \forall \phi \in C^2([0,1]).$$

Now if we choose 20 nodes from the subsets of $\mathbb{R}$ that time variable, state and control functions take their values in them, a linear programming with 8×10^3 variables as (13)-(14) concludes the approximate optimal control and trajectory which have been

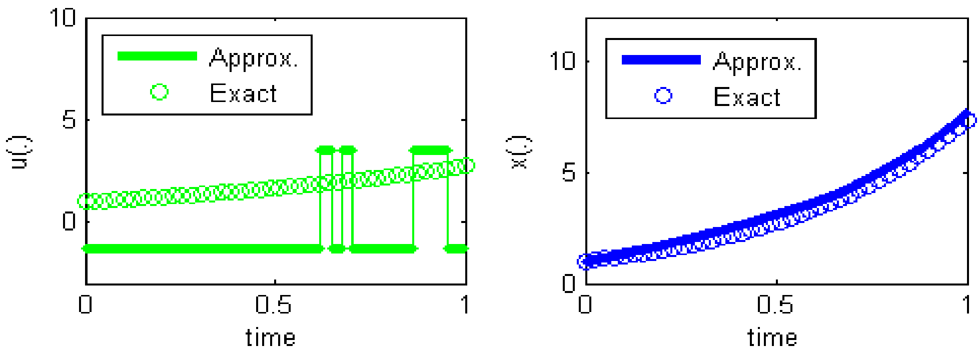

Figure 2: Exact and approximate optimal control and state functions in Example 2.

shown in Fig.1 and compared to exact solutions.

Example 2. In this example we consider the problem of minimizing the functional

$$\int_0^1 (x(t) - u^2(t))^2 dt$$

with the linear differential equation

$$tx'' + (1-2t)x' + tx = (2+t)u^2, \quad 0 < t < 1, \tag{16}$$

and the boundary conditions

$$x(0) = 1,\ x'(0) = 2,\ \ x(1) = e^2,\ x'(1) = 2e^2.$$

The optimal state and control function for this problem are $x^o(t) = e^{2t}$ and $u^o(t) = e^t$, respectively. A linear programming with 8×10^3 variables as (13)-(14) gives rise to the approximate optimal control and trajectory which have been shown and compared to exact solutions in Fig.2.

Example 3. For better showing the efficiency of the given method we consider the problem of minimizing the functional

$$\int_0^1 (x(t) - \sin t)^2 + (u(t) - t)^2 dt$$

governed by the linear differential equation

$$x^{(4)} + x^{(2)} + t^2 x = u^2 \sin t, \tag{17}$$

and the boundary conditions

$$x(0) = 0,\ x'(0) = 1,\ x^{(2)}(0) = 0,\ x^{(3)}(0) = -1$$

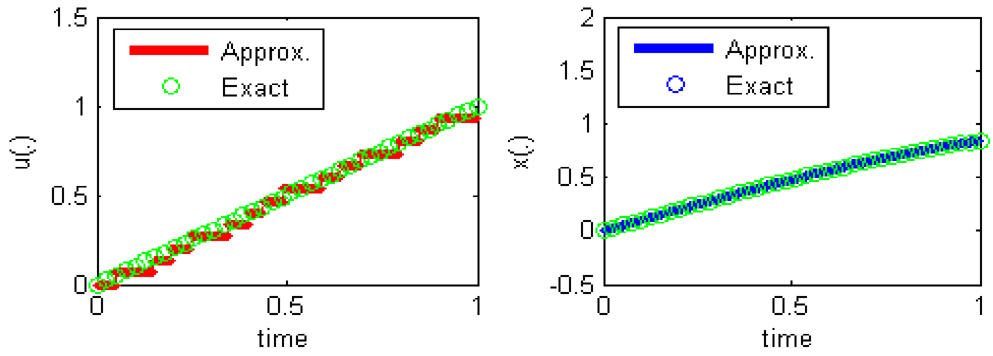

Figure 3: Exact and approximate optimal control and state functions in Example 3.

$$x(1) = \sin 1,\ x'(1) = \cos 1,\ x^{(2)}(1) = -\sin 1,\ x^{(3)}(1) = -\cos 1$$

The exact optimal state and control functions for this problem are $x^o(t) = \sin t$ and $u^o(t) = t$, respectively. Applying the usual measure theoretical approach needs to covert the differential equation (17) to a linear system with four state functions. Thus, choosing 20 nodes from the intervals of time, state and control functions give rise to a linear programming with 20^6 variables. The discussed approach which is to consider a metamorphosis on the problem of minimizing the above functional with the equivalent integral form of equation (17) that can be obtained by multiplying the test functions in (17) and integration by parts as

$$\int_0^1 ((\phi^{(4)} + \phi^{(2)} + t^2\phi)x - \phi u \sin t)dt = \alpha_\phi, \quad \forall \phi \in C^4([0,1])$$

where $\alpha_\phi = -2\phi(1)\cos 1 + 2\phi(0) - \phi^{(2)}(1)\cos 1 - \phi^{(2)}(0) - \phi^{(3)}(1)\sin 1$, give rise to the approximate optimal control and state functions that can be concluded from solving a linear programming with 20^3 variables. The comparison of exact and approximate control and state functions can be seen in Fig.3.

4 Conclusion

This article presents a different implementation of measure theoretical approach for finding approximate solutions in a class of optimal control problems. By some changes the number of variables in the linear programming which is obtained from metamorphosis and successive approximations is decreased dramatically. Of course there are many important issues that remain to be studied and which will be dealt with in a sequel to the present paper.

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