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Unique Continuation and Inverse Problems for Heat Equations

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Abstract

This paper is concerned with an inverse problem involving an N-dimensional heat equation having a non-constant thermal conductivity. Using a unique continuation result due to Saut and Scheurer we prove a uniqueness result.

Key words: Inverse problem, Heat equation, Unique continuation, Uniqueness

1 Introduction

Inverse problems arise have numerous applications in science and engineering. This paper is concerned with an inverse heat conduction problem, where the unknown thermal action on part of the boundary of the object is to be found based on observations (measurements) of the temperature in the interior. Therefore, the problem is classified as a *boundary* inverse problem. The heat equation considered describes the evolution of temperature in a medium Ω where the thermal conductivity p is a function of spatial variables and time. More precisely, we assume the domain of interest is divided into three regions Ω_1, $N(\Gamma, \Gamma_s)$ and Ω_2. The region $N(\Gamma, \Gamma_s)$ is confined between two parallel hypersurfaces Γ and Γ_s where the heat has been measured by sensors at each point *only at one time*. The datum in this region is represented by a function g which satisfies a technical condition given by (12). In addition, we assume that the heat measurement on the outer part of the boundary of Ω_1 is known and that of Ω_2 is *unknown*. This paper is not concerned with the determination of the unknown boundary

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datum but rather with the uniqueness. The main tool in our the analysis is a unique continuation theorem, recalled in Theorem 2 below, due to Saut and Scheurer [**6**]. This unique continuation result, applicable to time-dependent parabolic equations with variable coefficients which are not necessarily smooth, generalizes work of Mizohata [**4**]. The paper is organized as follows. In section two we give a precise description of the inverse problem and state the main result of the paper, Theorem 1 . In section three we present some preliminary lemmas. Finally in the last section we give the proof of Theorem 1.

2 Description of the inverse problem

In this section we give the description of the inverse problem.

2.1 Function spaces and the parabolic domain

Throughout the paper $p > N$, where $N \in \mathbb{N}$ denotes the dimension of the spatial variables. This technical condition makes the continuous embedding $W^{2,p}(O) \hookrightarrow C^{1,\alpha}(O)$, where O is a bounded smooth domain in $\mathbb{R}^N$, hold. Suppose for an interval (a,b), $X(a,b)$ denotes a function space consisting of real valued functions defined on (a,b). Similarly, for a bounded domain O in $\mathbb{R}^N$, $Y(O)$ is defined. The space $X(a,b;Y(O))$ denotes the set of functions $u(x,t)$ such that $u(\cdot,t) \in Y(O)$ and $u(x,\cdot) \in X(a,b)$. For $x \in \mathbb{R}^N$, we write $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, where $x' = (x_1 \cdots, x_{N-1})$. Let U be an open and bounded domain in $\mathbb{R}^{N-1}$. Let Γ denote a smooth hypersurface defined on U; that is, Γ is the graph of a smooth function $\gamma : U \to \mathbb{R}$. For $s > 0$, let Γ_s denote the hypersurface formed be translating Γ in the negative x_N-direction, so

$$\Gamma_s = \{(x', \gamma(x') - s) : \ x' \in U\}.$$

The space confined between Γ and Γ_s is designated by $N(\Gamma, \Gamma_s)$, called the *sensor location.* In other words

$$N(\Gamma, \Gamma_s) = \{x \in \mathbb{R}^N : \ x' \in U, x_N \in [\gamma(x') - s, \gamma(x')]\}.$$

Suppose that Ω_j, $j = 1, 2$, are two smooth bounded and open domains in $\mathbb{R}^N$; that is, Ω_j are smooth N-dimensional compact submanifolds in $\mathbb{R}^N$. We assume the following conditions are satisfied:
(ω_1) $\overline{\Omega}_1 \cap \overline{N}(\Gamma, \Gamma_s) = \Gamma$.
(ω_2) $\overline{\Omega}_2 \cap \overline{N}(\Gamma, \Gamma_s) = \Gamma_s$.
Here overline denotes the closure. We set $\Omega = \mathrm{int}(\Omega_1 \cup \Omega_2 \cup \overline{N}(\Gamma, \Gamma_s))$, where $\mathrm{int}(\cdot)$ denotes the interior of a set. Given $T > 0$, we set $\Omega_T = \Omega \times (0, T)$, the *parabolic domain.* Next we consider a smooth function $g : \overline{N}(\Gamma, \Gamma_s) \to [0, T]$ which satisfies the following conditions:
(g_1) $g^{-1}(0) = \Gamma$ and $g^{-1}(T) = \Gamma_s$.
(g_2) The graph of g has no flat sections; that is, sets of the form $g^{-1}(c)$, where $c \in [0, T]$, have N-dimensional Lebesgue measure zero.

(g_3) For any $t \in [0,T]$, the weak divergence theorem is applicable on the sets $\mathrm{int}(\Omega_1 \cup g^{-1}[0,t])$ and $\mathrm{int}(\Omega_2 \cup g^{-1}[t,T])$.
(g_4) For every $t \in [0,T)$, $g^{-1}[0,t]$ is connected.
(g_5) The distribution function of g, denoted λ_g, is continuously differentiable. Here

$$\lambda_g(\beta) = \mu_N(\{x \in N(\Gamma,\Gamma_s) :\ g(x) \geq \beta\}),$$

where μ_N denotes the N-dimensional Lebesgue measure.
(g_6) g satisfies the inequality (12), see the section 3.

2.2 The inverse problem

The notation introduced in the previous subsection are valid here as well.

Let us begin by considering a function $p \in C^1(\overline{\Omega}_T)$ satisfying

$$0 < p_m \leq p \leq p_M < \infty, \tag{1}$$

where p_m and p_M are constants. We denote by $L(\cdot)$ the heat operator $\frac{\partial}{\partial t}(\cdot) - \nabla \cdot (p(x,t)\nabla(\cdot))$.

Definition. We say $u \in H^1(0,T;W^{2,p}(\Omega))$ is a solution of $Lw = 0$, in $S \subseteq \Omega_T$, provided $Lu = 0$, almost everywhere in S.

We are now in position to state the inverse problem. Let us consider the following problem denoted (IP):

$$\begin{cases} Lw = 0, \ \ \text{in } \Omega_T \\ w(x,0) = f(x), \ \ x \in \Omega \\ w|_{g(\overline{N}(\Gamma,\Gamma_s))} = h_1(x,t), \ \ (x,t) \in \mathrm{graph}(g) \\ w(x,t) = h_2(x,t), \ \ (x,t) \in (\partial(\Omega_1 \cup N(\Gamma,\Gamma_s)) \setminus \Gamma_s) \times [0,T) \\ w(x,t) = \psi(x,t), \ \ (x,t) \in \partial\Omega_2 \setminus \Gamma \times [0,T), \end{cases}$$

where$f$ and h_j, $j = 1,2$, are given smooth functions; however *u and ψ* are *unknown.* Therefore (IP) is an *inverse problem.* In the following definition we present the precise meaning of a solution to (IP).

Definition. The pair $(u,\psi) \in H^1(0,T;W^{2,p}(\Omega)) \times C(\partial\Omega_2 \setminus \Gamma_s \times (0,T))$ is said to be a solution of (IP) provided
(H_1) u is a solution of $Lw = 0$, in Ω_T.
(H_2) (u,ψ) satisfies the remaining equations in (IP) in the sense of traces.

We are now in position to state the main result of the paper.

Theorem 1 *The inverse problem (IP) has at most one solution.*

3 Preliminaries

In this section we state and prove lemmas that are needed to obtain the main result. We begin by a function $u \in H^1(0,T;W^{2,p}(\Omega))$ and set

$$\eta(t) = \int_{\Omega_1 \cup g^{-1}([0,t])} u^2(x,t)\ dx.$$

We assume u is a solution to $Lu = 0$, in Ω_T^1. In addition we suppose $u(x,0) = 0$, for every $x \in \Omega_1$, and $u(x,t) = 0$, whenever $(x,t) \in \partial\Omega_T^1$. We claim that under certain technical condition, to be given later, it follows that $u \equiv 0$, in Ω_T^1. To prove this result we proceed as follows. Since $u \in H^1(0,T;W^{2,p}(\Omega))$, we infer, using standard embedding theorems that $u(.,t) \in C^{1,\nu}(\overline{\Omega})$, for every $t \in [0,T]$; also, $u(x,.) \in C[0,T]$, for every $x \in \Omega$. Therefore it follows that η is a continuous function on $[0,T]$. Denote the upper right derivative of η by $D^+\eta$; that is,

$$D^+\eta(t) = \limsup_{h\to 0^+} \frac{\eta(t+h)-\eta(t)}{h}.$$

Suppose for the moment that

$$D^+\eta(t) \leq 0. \tag{2}$$

Thus since η is continuous, it is a standard result that η is non-increasing on $[0,T]$. This in particular implies that $\eta(t) \leq \eta(0) = 0$. So since η is non-negative it follows that $\eta \equiv 0$, hence $u \equiv 0$, as claimed. Now we focus on proving (2). Let us write $\eta = \eta_1 + \eta_2$, where $\eta_1(t) = \int_{\Omega_1} u^2(x,t)\ dx$ and $\eta_2(t) = \int_{g^{-1}[0,t]} u^2(x,t)\ dx$. Note that $D^+\eta(t) \leq D^+\eta_1(t) + D^+\eta_2(t)$; moreover, $D^+\eta_1(t) = \eta_1'(t) = 2\int_{\Omega_1} uu_t\ dx$. We now find a suitable upper bound for $D^+\eta_2$. Fix $h > 0$. Then

$$\begin{aligned} \tfrac{\eta_2(t+h)-\eta_2(t)}{h} &= \tfrac{1}{h}\int_{g^{-1}[0,t+h]} (u^2(x,t+h) - u^2(x,t))\ dx \\ &+ \tfrac{1}{h}\int_{g^{-1}(t,t+h]} u^2(x,t)\ dx \end{aligned} \tag{3}$$

The second term on the RHS of (3) can be estimated as follows

$$\frac{1}{h}\int_{g^{-1}(t,t+h]} u^2(x,t)\ dx \leq \|u(.,t)\|^2_{L^\infty(g^{-1}(t,t+h])} \frac{\mu_N(g^{-1}(t,t+h])}{h}. \tag{4}$$

Note that

$$\|u(.,t)\|^2_{L^\infty(g^{-1}(t,t+h])} \leq C\,\|\nabla u(.,t)\|^2_{L^2(g^{-1}(t,t+h])}, \tag{5}$$

where C is a constant independent of t. Observe that

$$\mu_N(g^{-1}(t,t+h]) = \lambda_g(t) - \lambda_g(t+h). \tag{6}$$

Therefore incorporating (5) and (6) into (4) we derive

$$\frac{1}{h}\int_{g^{-1}(t,t+h]} u^2(x,t)\ dx \leq C\frac{\lambda_g(t)-\lambda_g(t+h)}{h}\,\|\nabla u(.,t)\|^2_{L^2(g^{-1}(t,t+h])}. \tag{7}$$

Therefore (7) and (3) yield

$$\begin{array}{rcl} \frac{\eta_2(t+h)-\eta_2(t)}{h} & \leq & \frac{1}{h}\int_{g^{-1}[0,t+h]}(u^2(x,t+h)-u^2(x,t))\ dx \\ & + & C\frac{\lambda_g(t)-\lambda_g(t+h)}{h}\left\|\nabla u(.,t)\right\|^2_{L^2(g^{-1}(t,t+h])} \end{array} \tag{8}$$

Hence

$$D^+\eta_2(t) \leq 2\int_{g^{-1}[0,t]} uu_t\ dx - C\lambda_g'(t)\left\|\nabla u(.,t)\right\|^2_{L^2(g^{-1}[0,t])}.$$

Thus

$$D^+\eta(t) \leq 2\int_{\Omega_1\cup g^{-1}[0,t]} uu_t\ dx - C\lambda_g'(t)\left\|\nabla u(.,t)\right\|^2_{L^2(g^{-1}[0,t])}. \tag{9}$$

On the other hand multiplying the differential equation by u and integrating over the set $\Omega_1\cup g^{-1}[0,t]$ yields

$$\int_{\Omega_1\cup g^{-1}[0,t]} uu_t\ dx = \int_{\Omega_1\cup g^{-1}[0,t]} u\nabla.(p\nabla u)\ dx. \tag{10}$$

Hence by applying the weak divergence theorem, see for example [**1**], to the right hand side of (10) we will get

$$\int_{\Omega_1\cup g^{-1}[0,t]} uu_t\ dx = -\int_{\Omega_1\cup g^{-1}[0,t]} p\left|\nabla u\right|^2\ dx. \tag{11}$$

Thus from (11) and (9) we find

$$D^+\eta(t) \leq -2\int_{\Omega_1\cup g^{-1}[0,t]} p\left|\nabla u\right|^2\ dx - C\lambda_g'(t)\left\|\nabla u(.,t)\right\|^2_{L^2(g^{-1}[0,t])}.$$

Now recalling the condition $p\geq p_m$, it follows that

$$D^+\eta(t) \leq (-2p_m - C\lambda_g'(t))\left\|\nabla u(.,t)\right\|^2_{L^2(\Omega_1\cup g^{-1}[0,t])}.$$

Therefore if we assume

$$-C\lambda_g'(t) \leq 2p_m, \tag{12}$$

then $D^+\eta(t)\leq 0$, as desired. We can now gather the above discussion into the following

Lemma 1 *Suppose $u\in H^1(0,T;W^{2,p}(\Omega))$ is a solution of the following problem*

$$\begin{array}{rcll} Lu & = & 0, & in\ \Omega_T^1 \\ u(x,0) & = & 0, & x\in\Omega_1 \\ u(x,t) & = & 0, & (x,t)\in\partial\Omega_T^1. \end{array}$$

In addition, suppose condition (12) holds. Then $u\equiv 0$, in Ω_T^1.

For the next lemma we use the notation $\tilde{\Omega} = \mathrm{int}(\Omega_2 \cup \overline{N}(\Gamma, \Gamma_s))$.

Lemma 2 *Let $T > 0$ and $u \in H^1(0,T; W^{2,p}(\Omega))$ is a solution of*

$$Lu = 0, \quad in\ \Omega_T^2. \tag{13}$$

Suppose also that

$$u(x,0) = 0, \quad x \in \tilde{\Omega} \tag{14}$$

and

$$u(x, g(x)) = 0, \quad \frac{\partial u}{\partial \nu}(x, g(x)) = 0, \quad x \in N(\Gamma, \Gamma_s), \tag{15}$$

where for $x \in N(\Gamma, \Gamma_s)$, $\nu(x)$ denotes the unit normal to the graph of g at the point $(x, g(x))$ pointing inside Ω_T^1. Set

$$w(x,t) = \begin{cases} u(x,t), & (x,t) \in \Omega_T^2 \\ 0, & (x,t) \in \tilde{\Omega} \times (-T,T) \setminus \Omega_T^2. \end{cases}$$

Then w is a distributional solution of $Lw = 0$, in $\tilde{\Omega} \times (-T,T)$; that is, for every $\xi \in C_0^\infty(\tilde{\Omega} \times (-T,T))$ the following integral equation holds

$$\int_{\tilde{\Omega} \times (-T,T)} w(\xi_t + \nabla \cdot (p \nabla \xi)) dx dt = 0. \tag{16}$$

Proof. From the definition of w it is clear that in order to show (16) we need only to show

$$\int_{\Omega_T^2} u(\xi_t + \nabla \cdot (p \nabla \xi)) dx dt = 0. \tag{17}$$

Let us first consider $\int_{\Omega_T^2} u \xi_t dx dt$. From the Cavalieri's principle we have

$$\int_{\Omega_T^2} u \xi_t dx dt = \int_{\Omega_2 \cup g^{-1}[0,t]} \int_0^T u \xi_t dt dx.$$

From (14) and the fact that ξ has compact support it follows that $\int_0^T u \xi_t dt = -\int_0^T u_t \xi dt$. Therefore

$$\int_{\Omega_T^2} u \xi_t dx dt = -\int_{\Omega_T^2} u_t \xi dx dt. \tag{18}$$

Next we consider $\int_{\Omega_T^2} u \nabla \cdot (p \nabla \xi) dx dt$. Writing this integral as $\int_0^T \int_{\Omega_2 \cup g^{-1}[0,t]} u \nabla \cdot (p \nabla \xi) dx dt$, we proceed by applying the weak divergence theorem to the inner integral. Thus we fix $t \in (0,T)$, hence

$$\int_{\Omega_2 \cup g^{-1}[0,t]} u \nabla \cdot (p \nabla \xi) dx = \int_{\partial(\Omega_2 \cup g^{-1}[0,t])} pu \frac{\partial \xi}{\partial \nu} d\sigma(x) - \int_{\Omega_2 \cup g^{-1}[0,t]} p \nabla u \cdot \nabla \xi dx. \tag{19}$$

From (15) we infer that the first integral in the RHS of (19) vanishes. Once again an application of the weak divergence theorem yields

$$\int_{\Omega_2 \cup g^{-1}[0,t]} p\nabla u \cdot \nabla \xi dx = \int_{\partial(\Omega_2 \cup g^{-1}[0,t])} p\xi \frac{\partial u}{\partial \nu} d\sigma(x) - \int_{\Omega_2 \cup g^{-1}[0,t]} \xi \nabla \cdot (p\nabla u) dx.$$

Therefore from the boundary condition (15) and (19) we derive

$$\int_{\Omega_2 \cup g^{-1}[0,t]} u\nabla \cdot (p\nabla \xi) dx = \int_{\Omega_2 \cup g^{-1}[0,t]} \xi \nabla \cdot (p\nabla u) dx.$$

Whence

$$\int_{\Omega_T^2} u\nabla \cdot (p\nabla \xi) dxdt = \int_{\Omega_T^2} \xi \nabla \cdot (p\nabla u) dxdt. \tag{20}$$

We obtain (17) from (18) and (20). $\diamondsuit$

The main tool in proving Theorem 1 is a unique continuation theorem, applied to a second order parabolic equation, due to Saut and Scheurer [**6**, Theorem 1.1]. The proof of Theorem 1.1 in [**6**], which is based on the derivation of a Carleman estimate which is reminiscent of the classical Carleman estimates for second order elliptic operators [**4**], is rather long so the reader is referred to the original paper for details. To state this result we need to give the definition of the horizontal component of an open set following Nirenberg [**5**].

Definition. Let $\mathcal{O}$ denote a connected open set in $\mathbb{R}^N \times \mathbb{R}$. Suppose that $\mathcal{O}_1$ is an open set contained in $\mathcal{O}$. Then the horizontal component of $\mathcal{O}_1$, denoted hor$(\mathcal{O}_1)$, is the union of all open segments $t =$ constant in $\mathcal{O}$ which contain a point of $\mathcal{O}_1$, hence

$$\{\text{hor}(\mathcal{O}_1) = \{(x,t) \in \mathcal{O} :\ \exists x_1,\ (x_1,t) \in \mathcal{O}_1\}.$$

We now state the unique continuation theorem in the framework of the present paper.

Theorem 2 *Let $\mathcal{Q}$ be a connected open set in $\mathbb{R}^N$ and $\mathcal{Q}_T = \mathcal{Q} \times (-T,T)$. Let $u \in L^2(-T,T;H^2_{loc}(\mathcal{Q}))$ be a solution of $Lw = 0$. In addition, we assume that u vanishes in an open set $\mathcal{Q}_1 \subseteq \mathcal{Q}$. Then u vanishes in hor$(\mathcal{Q}_1)$.*

4 Proof of Theorem 1

To prove Theorem 1, let us assume (u_1,ψ_1) and (u_2,ψ_2) are solutions of (IP). Setting $u = u_1 - u_2$, we find that

$$\begin{cases} Lu = 0, \ \text{ in } \Omega_T^1 \\ u(x,0) = 0, \ \ x \in \Omega_1 \\ u(x,t) = 0, \ \ (x,t) \in \partial\Omega_T^1. \end{cases}$$

Therefore, by Lemma 1, it follows that $u \equiv 0$, in Ω_T^1. Note that $u(\cdot,t) \in W^{2,p}(\Omega)$, so thanks to the embedding $W^{2,p}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$, it follows that

$u(\cdot,t) \in C^{1,\alpha}(\overline{\Omega})$. Hence $\frac{\partial u}{\partial \nu}(x, g(x))$, $x \in N(\Omega, \Omega_s)$, where $\nu(x)$ is as in Lemma 2, can be calculated in the ordinary sense. On the other hand since $u(x, g(x)) = 0$, $x \in N(\Gamma, \Gamma_s)$, we find that $\frac{\partial u}{\partial \nu}(x, g(x)) = 0$, $x \in N(\Gamma, \Gamma_s)$. Thus we can now apply Lemma 2 to deduce that w, as defined in Lemma 2, satisfies $Lw = 0$, in $\tilde{\Omega} \times (-T, T)$, in the sense of distributions. Whence by a standard regularity theory, see for example [**2**], it follows that $w \in H^1(-T, T; H^2_{loc}(\Omega))$. Finally, by applying the unique continuation, Theorem 2, we deduce $w \equiv 0$, in $\tilde{\Omega} \times (0, T)$. Hence $u \equiv 0$, in Ω^2_T. This obviously completes the proof of the theorem.

References

[1] P. Grisvard, "Singularities in boundary value problems," Springer-Verlag, Berlin, 1992.

[2] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Uralceva, "Linear and quasi-linear equations of parabolic type", Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, 1968.

[3] V.G. Mazja, "Sobolev spaces", Springer-Verlag, Berlin, 1985.

[4] S. Mizohata, *Unicitè du prolongement des solutions pour quelques opèrateurs diffèrentiels paraboliques*, Mem. Coll. Sci. Univ. Kyoto, Sèr. A 31(3), (1958), 219-239.

[5] L. Nirenberg, *Uniqueness in Cauchy problems for differential equations with constant leading coefficients*, Comm. Pure Appl. Math.,10 (1957), 89-105.

[6] J.C. Saut and B. Scheurer, *Unique continuation for some evolution equations*, J. Diff. Equ., (66)1, (1987), 118-139.