Analysis of Digraphs

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Abstract

We will shown some results about Directed Graphs and its equivalence classes, by the study of its enumerative asymptotic behaviour, according their different possibilities.

Keywords: Graph Theory, Combinatorics, Enumeration of graphs, Graph labeling, Asymptotic Analysis.

Mathematics Subject Classification: 68R10, 68R05, 05C78, 78M35.

1. Introduction

Graphs are mathematical objects frequently used in Computer Science, and other important fields. Because often we can reduce a real-world problem to a mathematical statement about graphs. Hence, if the graph problem is solved, then the real-world problem it is also solved.

A *graph* is a pair, $G = (V, E)$, where V is a set of points, called *nodes,* and E will be the set of their edges, or links, between nodes. Sometimes, it is denominated *simple graph.*

Note that a simple graph represents a symmetric relation, R, according which two nodes, a and b, being connected by an edge are related to each other in both directions, i.e. $a \ R \ b$ and $b \ R \ a$.

Many times, we said *Undirected Graph* (UG, in acronym), instead of simple graph.

V and E are usually taken to be *finite,* because some of the well-known results can fails for infinite graphs. It is due that many times the arguments are not applicable in such infinite case.

The *order* of a graph is the number of their nodes. That is, the cardinality of the set V, sometimes denoted by $|V|$.

The *size* of a graph is the number of their edges. That is, the cardinality of the set E, sometimes denoted by $|E|$.

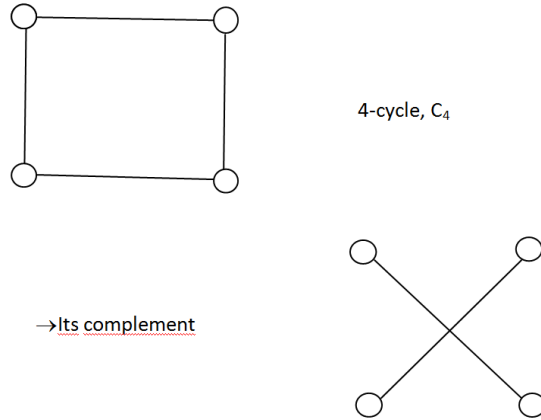The *degree* of a graph is the number of edges that connect to it.

Note that an edge that connects to the node at both ends (therefore, a loop) is counted twice.

Let $G = (V, E)$ be a graph. Then, we see as its *complement* the new graph

$$\overline{G} = \left(V, \overline{E}\right)$$

4-cycle, C_4

→Its complement

with the same set of nodes, but its set of edges contains the complement of E.

So, for instance, if we take the 4-cycle graph, C_4, its complement will be $\overline{C_4}$, according the representation that you can see in our first set of figures.

The set of edges of an undirected graph induce a symmetric binary relation on the set of its nodes. It is called the *adjacency relation* of such graph.

More formally, for each edge,

$$\{a, b\}$$

the nodes a and b are said to be *adjacent,* being denoted by

$$a \sim b$$

A directed graph, or abridgedly a *digraph,* is a graph in which each edge is replaced by a directed edge.

That is, from the set of unordered pairs of nodes, in which

$$\{a, b\} = \{b, a\}$$

we pass to the set of ordered pairs of nodes, where

$$(a, b) \neq (b, a)$$

Mathematically expressed,

$$E \subset V \ X \ V$$

being E the set of edges, and $V \ X \ V$ the Cartesian product of V with itself.

So, the definition of E coincides with the proposed for a relation on the set V. Therefore, a digraph can be used to model any relation on a set.

Note that the edges are regarded not as lines, but as arrows, going from a *tail* (or start) node to an *head* (or end) node.

A digraph having no multiple edges or loops is called a *simple digraph.*

And an *acyclic directed graph,* or *acyclic digraph,* also denoted by the acronym *DAG,* is a directed graph containing no directed cycles. Usually, their edges (directed) are called *arcs.*

A graph, G, is *transitive,* if given any three nodes,

$$a, \ b, \ c \in V \ (G)$$

if the edges $\{a, b\}$ and $\{b, c\}$ belongs to $E \ (G)$, then also the edge

$$\{a, c\} \in E \ (G)$$

An unlabeled n-digraph is called *a Topology.*

Let G be a graph. Its *Line Graph,* denoted by $L(G)$, has for node set the edge set of G, and also it has for edge set the node set of G.

I.e., they shown this interchange between the role of mathematical objects,

$$\{points\} \leftrightarrow \{lines\}$$

Or equivalently

$$\{nodes\} \leftrightarrow \{edges\}$$

and vice versa, between the graph and its line graph.

For instance, given the subsequent graph, G, with their edges labeled, then its line graph, L (G), would be as appears, according their incidence relationships as new edges, as you can see in our second set of figures.

There are many real-world phenomena which admits representation by graphs. For instance,

- *Computer Networks,* where the set of nodes represents the set of computers in the network.
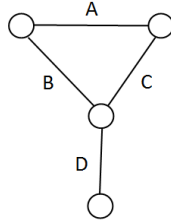
So, there is an edge if and only if there is a direct communication between the corresponding computers.

- *Program Flowchart,* where each node represents a step of computation, whereas the directed edges between pairs of nodes represent the control flow.
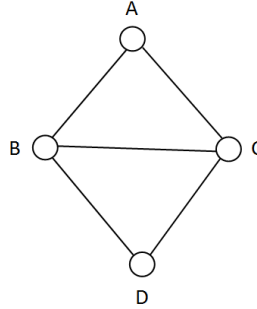
And many others applications, not only on Informatics, because it may be on

- *Airline Connections*

Graph G

Its Line Graph, L(G)

- *Two-Player Game Tree*

- *Precedence Constraints* (so, for jobs),

etc.

2. Enumerating Digraphs

Among the different graphs, *Bayesian Networks (BNs)* are a sucessful class of models to represent uncertain knowledge. But unfortunately, the representation of conditional independencies (CIs, in acronym) does not have uniqueness. The reason is that probabilistically equivalent models may have different representations.

This problem is overcome by the *Essential Graph,* as (unique) representant of each equivalence class. So, they represent such *CI models* by *graphs.*

Our mathematical tools contains both, directed or/and undirected edges.

Hence, producing either *Directed Graphs (DGs),* in particular with acyclic elements, i. e. *Directed Acyclic Graphs (DAGs),* or *Undirected Graphs (UGs).*

Finally, also *Chain Graphs (CGs),* i.e. the mixed case, with joint presence of both, directed and undirected edges.

So, *DAG models* are generally represented as *Essential Graphs (EGs).*

Then if we will known the behavior of the ratio among EGs to DAGs, we may dispose of a valuable tool in fields as Machine Learning, Artificial Vision, and so on. Because through this information, we may decide which space is the best to realize our search.

If the aforementioned ratio is low, we must prefer to search into the space of DAG models, rather than into the space of DAGs directly.

The most common approach to learning DAG models is that of performing such search into the space of either DAGs or DAG models (therefore, on EGs).

A very elegant auxiliary mathematical construct, if perhaps some difficult indeed, will be the Ihara Zeta function, devised for counting graphs.

Recall that a DAG, G, is *essential,* if every directed edge of G is protected. So, an *Essential Graph (EG)* is a graphical representation of a Markov equivalence class.

In relation with the Essential Graph, each directed edge would have the same direction in all the graphs that belongs to its equivalence class.

Hence, there is a bijective correspondence (one-to-one) among the set of Markov equivalence classes and the set of essential graphs, their representatives.

From the labeled or unlabeled character of the graph depends whether their nodes or edges are distinguishable or not.

For this, we will distinguish among *vertex-labeled, vertex-unlabeled, edge-labeled, or edge-unlabeled graphs.*

The labeling will be considered as a mathematical function (labeling), assigned to its elements, either nodes, or edges, or perhaps to both, making them so distinguishable.

Many times it will be useful to associate with each edge a certain number, called its *weight,* acting as label. For instance, it may be an estimation of the distance from the current node to the final node, in a graph of search. So, it can estimate the optimal number of moves from now until the checkmate, playing at chess.

In such way, we must to assign certain mastery of some link on another. Then, we call such mathematical constructs *edge-weighted,* or more simply, *weighted-graphs,* usually denoted by

$$w\left(a,\ b\right)$$

3. Searching for adequate bounds

It is possible to use generating functions to count labeled DAGs. For these, it is convenient to make intervene the *Inclusion-Exclusion Principle (IEP).*

So, if we take the set of *n-essential graphs,* and denote its cardinal by a_n, applying the *IEP,* we may obtain

$$a_n = \sum_{s=1,\ldots,p} (-1)^{s+1} \sum_{\substack{i_j \\ j\in\{1,\ldots,s\}}} c\left(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_s}\right)$$

where

$$A_k = \{G \in E : k \text{ is a } terminal \text{ } node \text{ of } G\},$$
$$\text{with } k = 1,\ 2,\ \ldots,\ n \quad [*]$$

Let a_n be the *number of essential (labeled) DAGs,* and also let $a_{n'}$ be the *number of (labeled) DAGs.*

Then, a_n is given by the recurrence equation

$$a_n = \sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s} - n + s\right)^{s} a_{n-s}$$

$$\text{with } a_0 = 1$$

Whereas we can obtain for the number of *labeled n-DAGs,*

$$a_n' = \sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^{s} a_{n-s}'$$

$$\text{with } a_0' = 1$$

We may rewrite the equation as

$$\sum_{s=0}^{n} (-1)^{n-s} C_{n,s} \left(2^{s} - s\right)^{n-s} a_s = 0$$

$$\text{with } n \geq 1.$$

Another case of application of *Inclusion-Exclusion Principle* is to find the cardinal of the set of essential DAGs, *E,* with a set of nodes

$$\{1, 2, \ldots, n\}$$

For this purpose, we may start with a family of sets, as the aforementioned

$$\{A_k\}_{k=1}^{n}$$

Therefore, to know the cardinal of *E,* firstly we must compute the intersection that appears in the last summatory, for $j = 1, 2, \ldots, n$. Showing the allowed connection numbers, from a given node being

$$2^{n-s} - n + s$$

So, the number of possible ways of adding directed edges from the essential graph until all the s terminal nodes will be

$$\left[2^{n-s} - n + s\right]^{s}$$

If we denote a_n the number of *essential* $n - DAGs$, this would be

$$a_n = \sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s} - (n - s)\right)^{s} a_{n-s}$$

$$\text{with } a_0 = 1$$

If the purpose were to obtain the number of *labeled n-DAGs,* it would be

$$a_n' = \sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^s a_{n-s}'$$

$$\text{with } a_0' = 1$$

Considering e_n the number of *essential n-graphs*, also *labeled*, it holds that

$$a_n \leq e_n \leq a_n'$$

I.e. both precedent values, a_n and a_n', are the lower and upper bounds of e_n, for each selected order, n.

So, it holds

$$\frac{1}{13.6} \leq \frac{a_n}{a_n'}$$

Therefore,

$$\frac{1}{13.6} a_n' \leq e_n \leq a_n'$$

where we obtain the bounds for the cardinality of the set of essential graphs.

3. Riemann Zeta Function

Among the Dirichlet Series, we found a very useful tool.

It is the so called *Riemann Zeta Function,*

$$\varsigma(s) = \sum_{n \in \mathbf{N}} \frac{1}{n^s} = \prod_{p \ prime} \left(1 - p^{-s}\right)^{-1}$$

$$\text{with } s \in \mathbb{C}$$

Where it appears as an Euler product.

And because the bounded sequence of coefficients, these series converge absolutely to an analytical function, on the complex open half-plane of s such that

$$Re(s) > 1$$

It diverges on the symmetrical open half-plane of s, in the complex plane,

$$Re(s) < 1$$

Such function, if it is defined on the first region, admits analytic continuation to all $\mathbb{C}$, except when $s = 1$.

For $s = 1$, this series is formally identical to the Harmonic series, which is well-known that diverges.

As a consequence, it will be a meromorphic function of s, being in particular, holomorphic in a region of the complex plane, showing one pole in $s = 1$, with residue equal to 1.

Euler found a closed formula for $\zeta(2k)$, when k non-negative integer number.

It will be expressed by

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}B_{2k}}{2\,(2k)!}$$

denoting by B_{2k} the *Bernoulli numbers.*

Such numbers can be defined of different modes. So, either

- as independent terms of *Bernoulli polynomials,* $B_n(x)$,

- by a generating function, exactly

$$G(x) = \frac{x}{e^x - 1}$$

being so

$$G(x) = \sum_{i \in N^*} B_n \frac{x^i}{i!}$$

when

$$|x| < 2\pi$$

where each coefficient of the Taylor Series would be the *n-th Bernoulli number,*

or

- by a recursive formula

$$B_0 = 1$$

$$B_m = -\sum_{j=0}^{m-1} C_{m,j} \frac{B_j}{m-j+1}$$

But as the Bernoulli numbers can be expressed in terms of the Riemann Zeta function, they are indeed values of such function to negative arguments.

Some of the *Zeta Function special values* are

$$\varsigma(0) = -1/2$$

$$\varsigma(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$$

$$\varsigma(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + ... = \frac{\pi^4}{90}$$

$$\varsigma(6) = 1 + \frac{1}{2^6} + \frac{1}{3^6} + ... = \frac{\pi^6}{945}$$

$$\varsigma(8) = 1 + \frac{1}{2^8} + \frac{1}{3^8} + ... = \frac{\pi^8}{9450}$$

$$...$$

Note that we take here s *even.* Because for *odd* values of s, it appears troubles and also irrational numbers; so, for instance,

$$\varsigma\left(1\right) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots \rightarrow \infty$$
$$(harmonic \ \text{series})$$

$$\varsigma\left(3\right) = 1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \ldots \simeq 1.2$$
$$(Apéry \ constant)$$

$$and \ also$$
$$\varsigma\left(1/2\right) \simeq -1.46$$
$$\varsigma\left(3/2\right) \simeq 2.6$$
$$\varsigma\left(5/2\right) \simeq 0.134$$
$$\varsigma\left(7/2\right) \simeq 1.127$$
$$\ldots$$

The *Logarithm of the Zeta Function* will be

$$\log \varsigma\left(s\right) = \sum_{n \geq 2} \left(\frac{\Lambda(n)}{\log \ n}\right)\left(\frac{1}{n^s}\right)$$

$$being$$
$$\mathrm{Re}\left(s\right) > 1$$

Here, $\Lambda\left(n\right)$ denote the *Lambda Function,* also called sometimes *Von Mangoldt function,* defined by

$$\Lambda\left(n\right) = \begin{cases} \log \ p, \ if \ n = p^k, \ for \ n \in N \ and \ some \ prime \ number, \ p \\ 0, \ otherwise \end{cases}$$

It is an arithmetic function that is neither additive nor multiplicative,

$$\Lambda\left(n + n'\right) \neq \Lambda\left(n\right) + \Lambda\left(n'\right)$$
$$\Lambda\left(n \cdot n'\right) \neq \Lambda\left(n\right) \cdot \Lambda\left(n'\right)$$

Such Lambda function satisfies

$$\log \ n = \sum_{d|n} \Lambda\left(d\right)$$

where the summation will be extended to all integers, *d,* dividing to *n.*

Related with the above series, we have the popular *Riemann Hypothesis,* still an important open problem in current Mathematics. It is about the distribution of zeroes of such Zeta Function.

It admits many variations, with different names, Selberg, Ihara, etc. So, for instance, we may consider its multiplicative inverse, expressible as a series by the *Möbius Function.* It can be reached, from the known series, by tools as the *Möbius Inversion* and the *Dirichlet Convolution.* The values produced by such function from integer arguments are called *"zeta constants".*

We can observe their convergence to one from the right,

$$\varsigma\left(s\right) \to 1^{+}$$

Also, this *functional equation* is satisfied

$$\varsigma\left(s\right) = 2^{s}\,\pi^{s-1}\,\sin\left(\tfrac{\pi s}{2}\right)\,\Gamma\left(1-s\right)\,\varsigma\left(1-s\right)$$

which is true in all the complex field, relating its values in s and $1-s$.

This equation has a pole simple at s = 1, with residuum equal to one. It was proved by Riemannn (1859).

Euler conjectured an equivalent relation to the function

$$\sum_{n\in N^{*}}\frac{(-1)^{n+1}}{n^{s}}$$

Also there exists a symmetric version of the precedent functional equation, reachable by the change

$$s \longmapsto 1-s$$

This gives an important equation relating both, Zeta and Gamma functions.

$$\varsigma\left(s\right)\,\Gamma\left(\tfrac{s}{2}\right)\,\pi^{-\frac{s}{2}} = 2^{s}\,\pi^{-\frac{1-s}{2}}\,\Gamma\left(\tfrac{1-s}{2}\right)\,\varsigma\left(1-s\right)$$

The value of the Zeta function for negative even real values is then zero,

$$\varsigma\left(-2\right) = \varsigma\left(-4\right) = \varsigma\left(-6\right) = \;...\; = \varsigma\left(-2k\right) = 0$$
$$with\ k \in \mathbf{N}$$

They are called *trivial zeroes of ζ*.

Furthermore, it will be cancelled on values of s that belongs to the *critic rang*

$$\{s \in C : 0 < \operatorname{Re}\left(s\right) < 1\}$$

which correspond to a certain vertical strip of width equal to one, into the complex plane.

In this case, we call of *non-trivial zeroes.* It is so called because the difficulties to find its position into the critical rang.

To obtain zeta function values for some negative and non integer argument, we may proceed by

$$\varsigma\left(-1/2\right) = \frac{2^{-3/2}\pi^{-1/2}\Gamma(-1/2)}{\sin\left(\frac{\pi s}{2}\right)\varsigma(1-s)} \simeq \frac{-4\pi}{2.6} \simeq -0.2069$$

where Γ represents the *Gamma Function of Euler.*

Such functional equation also gives an asymptotic limit, proposed by (Nemes, 2007); exactly,

$$\varsigma\left(1-s\right) = \left(\tfrac{s}{2\pi l}\right)^{s} \sqrt{\tfrac{8\pi}{s}}\cos\left(\tfrac{\pi s}{2}\right)\left(1 + O\left(\tfrac{1}{s}\right)\right)$$

4. Ihara Zeta Function

The *Zeta Function* is *generalizable to graphs,* according to the theory elaborated by (Ihara, 1966). This function was first defined in terms of discrete subgroups.

J. P. Serre suggested can be reinterpreted graph-theoretically.

It was (Sunada, 1985) who put this into practice. In this version, it is usually denoted by ς_G, and defined by

$$\varsigma_G\left(s\right) \equiv \left[\prod_p \left(1 - s^{L(p)}\right)\right]^{-1}$$

This product is taken over all the primes, p, on the graph G, being $L(p)$ the length of the *prime p.*

The *primes in graphs* are equivalence classes of closed backtrackless tailless primitive paths. As a good example, you can observe the third figure, where we have different primes, as may be

$$[C] = [E_1,\ E_2,\ E_3]$$
$$[D] = [E_4,\ E_5,\ E_3]$$

and from them, by concatenation, or juxtaposition, we obtain a new prime,

$$[Y] \equiv [C\ D] = [E_1,\ E_2,\ E_3,\ E_4,\ E_5,\ E_3]$$

Also ς_G is always representable as the reciprocal of a polynomial

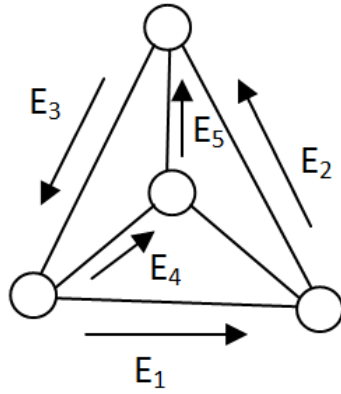$$\varsigma_G\left(s\right) \equiv \tfrac{1}{\det\ (I\ -\ T\ s)}$$

where T is the *edge adjacency operator* (Hashimoto, 1990).

For any *Graph, G,* the function ς_G can be expressed in terms of ς, for different dimension values, n.

So,

If $n = 1$, *then*

$$\varsigma_G\left(s\right) = 2\varsigma\left(s\right)$$

If $n = 2$, then

$$\varsigma_G(s) = 4\varsigma(s-1)$$

If $n = 3$, then

$$\varsigma_G(s) = 4\varsigma(s-2) + 2\varsigma(s)$$

If $n = \infty$, then

$$\varsigma_G(s) = \tfrac{8}{3}\varsigma(s-3) + \tfrac{16}{3}\varsigma(s-1)$$

Let G be a graph, and

$$A \equiv \left(a_{ij}\right)$$

its adjacency matrix, which will be a

$$(c\{V(G)\} \ x \ c\{V(G)\}) - matrix$$

with respective entries

$$a_{ij} \equiv \begin{cases} cardinal \ of \ undirected \ edges \ connecting \ n_i \ to \ n_j, \ being \ i \neq j \\ double \ of \ the \ cardinal \ of \ loops \ at \ the \ node \ n_i, \ if \ i = j \end{cases}$$

As our graphs have no loops neither multiple edges, such entries will be either zero or one, according to the adjacency or not adjacency of its respective pairs of nodes.

Suppose that we take now D, as the diagonal matrix such that its entry d_i is the degree of the i-th node minus one, and let

$$r - 1 = c\{E(G)\} - c\{V(G)\}$$

Then, the Ihara zeta function will be expressed as

$$\zeta_G(s)^{-1} \equiv \left(1 - s^2\right)^{r-1} \det\left(I - As + Du^2\right) =$$

$$= \frac{\det\left(I - A\ s + D\ u^2\right)}{\left(1 - s^2\right)^{1-r}}$$

It will be very interesting to look at the logarithmic derivative of the Ihara zeta function,

$$s\frac{d}{ds}\ln\left[\zeta_G(s)\right]$$

We have

$$\ln\left[\zeta_G(s)\right] = \ln\left[\prod_p\left(1 - s^{L(p)}\right)\right]^{-1} =$$

$$= \ln\left[\prod_p\left(1 - s^{L(p)}\right)^{-1}\right] = -\sum_p \ln\left(1 - s^{L(p)}\right)$$

and taking its derivative,

$$\frac{d}{ds}\left[\zeta_G(s)\right] = \frac{d}{ds}\left[-\sum_p \ln\left(1 - s^{L(p)}\right)\right] = -\sum_p \frac{1}{1-s^{L(p)}}\frac{d}{ds}\left(1 - s^{L(p)}\right) =$$

$$= -\sum_p \frac{1}{1-s^{L(p)}}\left[-L(p)\right]\ L(p) = \sum_p \frac{L(p)\ s^{L(p)-1}}{1-s^{L(p)}}$$

and now multiplying by s,

$$s\frac{d}{ds}\left[\zeta_G(s)\right] = s\sum_p \frac{L(p)\ s^{L(p)-1}}{1-s^{L(p)}} = \sum_p \frac{L(p)\ s^{L(p)}}{1-s^{L(p)}}$$

But such expression may be notably improved by the *geometric series* identity,

$$\sum_{n\in\mathbf{N}^*} s^n = \frac{1}{1-s}$$

giving

$$s\frac{d}{ds}\left[\zeta_G\left(s\right)\right] = \sum_p L\left(p\right)\,s^{L(p)}\left[1 + s^{L(p)} + s^{2L(p)} + s^{3L(p)} + \ldots\right] =$$

$$= \sum_p L\left(p\right)\left[s^{L(p)} + s^{2L(p)} + s^{3L(p)} + s^{4L(p)} + \ldots\right]$$

If we denote

$$N_k = \sum_{p:\ L(p)\,|\,k} L\left(p\right)$$

it holds

$$s\frac{d}{ds}\left[\zeta_G\left(s\right)\right] = \sum_{k\in\mathbf{N}} N_k\,s^k$$

Where the coefficient N_k, being associated with the term s^k, will report us the number of prime paths with a number of nodes which divides k.

Recall that $\varsigma_G\left(s\right)$ is a *decreasing function* of s. That is,

$$\varsigma_G\left(s_1\right) > \varsigma_G\left(s_2\right),$$

$$if\ s_1 < s_2$$

And in the limit situation, that is, if $n \to \infty$, when s is next to the *transition point,* it holds

$$\varsigma_G\left(s\right) = \frac{2^n\,\varsigma(s-n+1)}{\Gamma(n)}$$

If the average degree of nodes, also called *mean coordination number of the graph,* is finite, then there exists exactly a value of s, denoted $s_{transition}$, where the Zeta Function changes from infinite to finite, or vice versa.

5. Asymptotic behavior

Analyzing the asymptotic behavior of the ratio, i.e. studying the convergence of ratios among the number of classes, or essential digraphs, and the number of DAGs, we may develop

$$A\left(n\right) \equiv \frac{a_n}{a_n'} \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty} A\left(n\right) = \lim_{n\to\infty} \frac{\sum_{s=1}^{n}(-1)^{s+1}C_{n,s}\left(2^{n-s}-(n-s)\right)^s a_{n-s}}{\sum_{s=1}^{n}(-1)^{s+1}C_{n,s}\left(2^{n-s}\right)^s a_{n-s}'} =$$

$$= \lim_{n\to\infty} \frac{\sum_{s=1}^{n}C_{n,s}\left(2^{n-s}-(n-s)\right)^s a_{n-s}}{\sum_{s=1}^{n}C_{n,s}\left(2^{n-s}\right)^s a_{n-s}'} =$$

$$= \lim_{n \to \infty} \frac{\sum_{s=1}^{n} \left(2^{n-s} - (n-s)\right)^{s} a_{n-s}}{\sum_{s=1}^{n} \left(2^{n-s}\right)^{s} a_{n-s}{}'} =$$

$$= \lim_{n \to \infty} \frac{\left(2^{n-s} - (n-s)\right)^{s} a_{n-s}}{\left(2^{n-s}\right)^{s} a_{n-s}{}'}$$

But if we denote

$$A(n-s) \equiv \frac{a_{n-s}}{a_{n-s}{}'}$$

and

$$\varsigma_G (n-s) \equiv \lim_{n \to \infty} \sum_{s=1}^{n} \left(\frac{(n-s)}{2^{n-s}}\right)^{s}$$

Returning to our initial step,

$$\lim_{n \to \infty} A(n) = \lim_{n \to \infty} \left\{ \left[1 - \varsigma_G (n-s)\right] A(n-s) \right\} =$$

$$= \left[1 - \lim_{n \to \infty} \varsigma_G (n-s)\right] \left[\lim_{n \to \infty} A(n-s)\right]$$

Considering the partial sums

$$\sum_{s=1}^{n} \frac{n-s}{2^{n-s}}$$

and according to its asymptotical behaviour, when n increase, i.e. when $n \to \infty$, there is a correspondence with the so called *Ihara-Selberg of the n-graph* G_n.

Because operating here on the increasing value of $n - s$, i.e. with

$$\chi_G (n-s)$$

and denoting the ratio among terms of the series as

$$c = \lim_{n \to \infty} \frac{\left\{\sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s} - (n-s)\right)^{s} a_{n-s}\right\}}{\left\{\sum_{s=1}^{n} (-1)^{s+1} C_{n,s} \left(2^{n-s}\right)^{s} a_{n-s}\right\}}$$

it holds

$$c = \left(\lim_{n \to \infty} \left\{1 - \varsigma_G (n-s)\right\}\right) \left(\lim_{n \to \infty} A(n-s)\right)$$

As

$$a_{n-s} \leq a'_{n-s}, \forall n, \ \textit{once fixed } s \Rightarrow$$

$$\Rightarrow \exists \Phi_1 = \left[1 - \lim_{n \to \infty} \varsigma_G (n-s)\right] \left[\lim_{n \to \infty} A(n-s)\right]$$

once *fixed s, when n increases to* ∞,

$$\Phi_1 = \frac{1}{10 \ \varsigma\left(\frac{5}{2}\right)} = \frac{1}{5 \varsigma_G\left(\frac{5}{2}\right)}$$

equivalence classes for each digraph, or equivalently,

$$\Phi_1^{-1} = 10 \ \varsigma\left(\tfrac{5}{2}\right) = \tfrac{5}{2} \ \varsigma_G\left(\tfrac{7}{2}\right)$$

digraphs for each equivalence class.

In the bidimensional case (denoted by star notation), it holds

$$\exists \Phi_2^* \equiv \lim_{n \to \infty} \left(\left\{\varsigma_G (n)\right\} \left[A^*(n-s)\right]\right) \Rightarrow$$

$$\Rightarrow \Phi_2^* = \tfrac{1}{10} \ \varsigma\left(\tfrac{3}{2}\right) = \tfrac{1}{40}\varsigma_G\left(\tfrac{5}{2}\right)$$

and so dually,

$$\exists \left(\Phi_2^*\right)^{-1} \equiv \lim_{n \to \infty} \left(\left\{\varsigma_G (n)\right\} \left[A^*(n-s)\right]\right)^{-1} \Rightarrow$$

$$\Rightarrow \left(\Phi_2^*\right)^{-1} = \frac{10}{\varsigma\left(\frac{3}{2}\right)} = \frac{4}{\varsigma_G\left(\frac{5}{2}\right)}$$

6. Conclusion

So, finally, we can observe such behavior in its limit, reflecting the degree of fitness among the proposed construct, or model, and the "real" situation.

From now, we may to take subsequent steps towards a more powerful analytical framework, which permits to improve our theoretical basis, attempting to maintain the coherence with all these precedent results.

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