Fuzzifying Asymmetries

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Abstract

If we define Symmetry as a continuous feature, we will naturally get a more complex definition than the discrete one, but more useful in some fields, as Computer Vision. Therefore, the interest of such a definition is not only theoretical, but also applied, for instance in A. I. It will be very convenient to introduce "shade regions", when we analyze the Asymmetry of Shapes, modulating their degree of symmetry. We are so aiming at a fuzzy concept.

In this paper, we analyze the Asymmetry problem fundamentally, in its geometrical basis. We start with Shape Measures, Chirality Measure and so on. Then, searching for more efficient tools, until reaching the Asymmetry Level Function, as a new Normal Fuzzy Measure, with the proof of two new theorems about it, and some corollaries from them.

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1 Symmetry

We may attempt three ways for climbing the summit of asymmetry measures. First, the geometrical characterization of Symmetry, through group theory tools [10, 18].

Second, by statistical machinery, through distribution or density, and characteristic functions for instance, measuring the symmetry degree and the skewness of different probability distributions [13].

And third, by applying Measure Theory [16], in its more recent and adequate fuzzy version $[2-7,\ 17]$.

In this way, the distance from Symmetry of a shape is quantified as a continuous feature, instead of a discrete one: not only total coincidence and absolute difference are considered, but also gradual distance from its symmetrical shape $[2-7]$.

The Fuzzy version of problems [1] is now very useful indeed. Because taking into account the Entropy concept, a clue from Information Theory, the Fuzzy

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Variables can be transformed into Random Variables, and vice versa. So, the Fuzzy Structure would be considered as a Random Structure, and vice versa. In fact, it is a particular case, within the aforementioned mathematical construct.

Recall that both Uncertainty Measures play an ever important role in both general A. I. and Reasoning under Uncertainty. For instance, it is almost impossible to avoid dealing with Uncertainty in fields as Decision Making.

2 Shape measures

Our purpose in this section is to prepare the way for some measures of asymmetry and skewness, that is going to be useful when we work, for instance, with probabilistic distributions. It is possible to classify, within a determinate standard distribution, its variations respect to the model chosen as totally symmetrical.

We analyze the Symmetry as related to the more general case, that is, multivariate probability distributions. The univariate is just a simple particular case.

Let $\mathbf{X} = (X_1,\ X_2, ...,\ X_n) \in R^n$ be a random vector.

And let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \in R^n$ represent mean, mode or median, well-known centralization measures of the distribution.

So,

$$\mathbf{X} - \boldsymbol{\alpha} = (X_1 - \alpha_1,\ X_2 - \alpha_2, ...,\ X_n - \alpha_n) \in \mathbf{R}^n$$

Therefore, they are three n-dimensional vectors.

There exist many examples of multivariate symmetry, according to the invariance of such "centered" random vector, $\mathbf{X} - \boldsymbol{\alpha}$, under an appropriate family of transformations.

For instance, *spherical, elliptical, central* and *angular symmetry* (in increasing order of generality).

A random vector, $\mathbf{X}$, shows a *spherically symmetric* distribution about $\boldsymbol{\alpha}$, if rotation around $\boldsymbol{\alpha}$ does not alter the distribution.

So:

$$\mathbf{X} - \boldsymbol{\alpha} = \mathbf{A}\ \ (\mathbf{X} - \boldsymbol{\alpha})$$

where $\mathbf{A}$ represents any $(n \times n) -$ dimensional orthogonal matrix.

The sign $=$ signifies equality in distribution.

It is possible to characterize such spherical symmetry: it would be when $\parallel \mathbf{X} - \boldsymbol{\alpha} \parallel$ and $\frac{\mathbf{X}-\boldsymbol{\alpha}}{\parallel\mathbf{X}-\boldsymbol{\alpha}\parallel}$ are independent, being $\parallel . \parallel$ the Euclidean Norm, and $\frac{\mathbf{X}-\boldsymbol{\alpha}}{\parallel\mathbf{X}-\boldsymbol{\alpha}\parallel}$ uniformly distributed on the unit sphere surface of $\mathbf{R}^n$, denoted S^{n-1}.

It may also be characterized through the projection of $\mathbf{X}-\boldsymbol{\alpha}$ onto lines through the origin. They must have identical distributions.

Or also by the intervention of half-spaces.

The random vector $\mathbf{X}$ has a:

elliptically symmetric
(also called *ellipsoidally symmetric*)

distribution of parameters α and Σ, if it is affinely equivalent to that of a spherically random vector $\mathbf{Y}$, that is:

$$\mathbf{X} = \mathbf{A'} \cdot \mathbf{Y} + \boldsymbol{\alpha}$$

denoting $\cdot$ the matricial product, and satisfying:

$$\mathbf{A'} \cdot \mathbf{A} = \boldsymbol{\Sigma}$$

The name comes from the distribution being elliptically symmetric, then the contours of equal density are elliptical in shape.

The class of elliptically symmetric distributions is closed under conditioning, and also under affine transformations.

The random vector $\mathbf{X}$ has a *centrally symmetric distribution about* α, if:

$$\mathbf{X}-\boldsymbol{\alpha} = \boldsymbol{\alpha}-\mathbf{X}$$

Relaxing the notion of "central symmetry", we obtain that of "degree of symmetry".

Let $\mathbf{C}_k$ be a $(n \times n) - matrix,$ defined by:

$$\mathbf{C}_k = (c_{ij})_k = \begin{cases} -1, & if\ i = j = k \\ 0, & if\ i \neq j \\ +1, & if\ i = j \neq k \end{cases}$$

A random vector, $\mathbf{X}$, is said symmetric of degree m, if there exists a vector:

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_m, 0, 0, ..., 0)' \in \mathbf{R}^n$$

and a orthogonal transformation, T, such that:

$$T\ \ (\mathbf{X}-\boldsymbol{\alpha}) = C_1 \cdot C_2 \cdot ... \cdot C_m\ \ [T\ \ (\mathbf{X}-\boldsymbol{\alpha})]$$

This means that the distribution shows symmetries about m mutually orthogonal $(n-1)-$dimensional hyperplanes. Hence, also about its $(n-m)-$dimensional intersections, so it posesses m orthogonal directions of symmetry.

And now, around the angular symmetry of the random vector $\mathbf{X}$ about α,if:

$$\frac{\mathbf{X}-\boldsymbol{\alpha}}{\|\mathbf{X}-\boldsymbol{\alpha}\|} = \frac{\boldsymbol{\alpha}-\mathbf{X}}{\|\mathbf{X}-\boldsymbol{\alpha}\|}$$

Or in an equivalent expression:

if $\frac{\mathbf{X}-\boldsymbol{\alpha}}{\|\mathbf{X}-\boldsymbol{\alpha}\|}$ has centrally symmetric distribution.

3 Chirality Measure

The first question coming to mind is about its name:

What is Chirality?

Let us start with a well known quotation of Lord Kelvin [11]:

"I call any geometrical figure, or group of points, *chiral,* and say that it has *chirality,* if its image, in a plane mirror, ideally realized, cannot be brought to coincide with itself".

This opinion is supported on the classic and dichotomous division: totally symmetric versus totally asymmetric, without intermediate terms, into an Euclidean set.

A system is called *chiral*, if it differs from its mirror image, and such mirror image cannot be superposed on the original system. It is the famous case of our hands, our ears, and so on: it is impossible to superimpose with total coincidence our left hand over the mirror image of our right hand. For this reason, we need two *different* gloves, in order to cover our hands.

Therefore, we say that an object is *Chiral* when it is non-isomorphic to its mirror-image. Its symmetry group only contains pure translations, pure rotations, and also screw rotations.

When a system or object is not chiral, we says that is *achiral.*

Both elements of the pair (original chiral object, its mirror image) are denominated mutually *Enantiomorphs,* from the old greek *"opposite forms".*

Their mutual relation is an *enantiomorphism.*

When it refers to molecules, we says *enantiomers.*

The degree of such feature is measured by the *Chiral Index* (here denoted *Chi*).

In the univariate case, it is expressed from the lower bound of the correlation coefficient (ρ):

$$R_{\min} = lower\ bound\ \rho$$

between the distribution and itself.

Its mathematical expression will be:

$$Chi = \frac{1+R_{\min}}{2}$$

At previous step, we must suppose the existence of both statistical parameters: variance and mean.

The range of this function will be:

$$\Re_{Chi} \in [0,\ 0.5]$$

Obviously, if the object is *Achiral* (A)*,* then its index is null:

$$Chi\,(A) = 0$$

Furthermore, this property of symmetry is very important in many scientific fields, as for instance, studying the geometry of the molecular structure in chemical compounds.

It is possible to define such $Chirality\ Measure$ for a space having any dimension, for which probability distributions may be very useful. Recall that considering the n-dimensional Euclidean space, *a finite number of equally weighted points can be considered as a n-dimensional distribution.*

Two basic aspects are necessary. First, the *Chiral Index* may be *invariant under isometric transformations* applied on the probability distribution. And second, it may be *independent of* which particular *mirror* we have *selected.*

An achiral object may be superimposed on its mirror image, and then, its symmetry group possesses certain operations inverting its geometry, as can be glide reflections, not being this possible by a direct movement of a rigid body.

4 Fuzzy Measures

Recall some necessary definitions from Fuzzy Measure Theory.

Definition 1 *Let* U *be the universe of discourse, with* $\wp$ *a* $\sigma-$*algebra on* U*.*
Then, given a function
$m : \wp \rightarrow [0,\ 1]$
we describe m *as a Fuzzy Measure, if it verifies:*

I) $m\ (\emptyset) = 0$

II) $m\ (U)\ = 1$

III) If $A,\ B\ \in \wp$*, with* $A \subseteq B \Rightarrow m\ \ (A) \leq m\ \ (B)$ *[monotonicity]*

When we take the Entropy concept, we attempt to measure the fuzziness, that is, the degree of being fuzzy for each element in $\wp$*.*

Definition 2 *The Entropy can be designed as the function:*
$H : \wp \rightarrow [0,\ 1]$
verifying:

I) If A *is a crisp set* $\Rightarrow H\,(A) = 0$

II) If $H\ \ (x) = 1/2,\ \forall x \in A \Rightarrow H\ \ (A)$ *is maximal (total uncertainty)*

III) If A *is less fuzzified than* $B \Rightarrow H\ \ (A) \leq H\ \ (B)$

IV) $H\ \ (A) = H\ \ (U \setminus A)$

And also:

Definition 3 *The Specificity Measure will be introduced as a measure of the tranquillity when we take decisions.*

Such Specificity Measure is a function:

$S\ p : [0,\ 1]^U \to [0,\ 1]$

where:

I) $S\ p\ (\emptyset) = 0$

II) $S\ p\ (\varkappa) = 1 \Leftrightarrow \varkappa$ *is a unitary set (singleton)*

III) If ς *and* τ *are normal fuzzy sets in* U

with: $\varsigma \subset \tau \Rightarrow S\ p\ (\varsigma) \geq S\ p\ (\tau)$.

Note: $[0,\ 1]^U$ denotes the class of fuzzy sets in U.

5 Assymetry/Symmetry Measures

Let *(E, d)* be a fuzzy metric space.

Note: our results may also be generalized to other spaces.

We proceed to define the new fuzzy measure, which is a new and useful function.

Such application would be defined as one of the kind:

$$L_i$$

with $i \in \{a,\ s\}$, where s denotes *symmetry*, and a denotes *asymmetry.*

From here onwards, we denote by $c(A)$ the cardinal of a fuzzy set A.

Theorem 1 *Let* (E, d) *be a fuzzy metric space, being* A *a subset of* E*, and let* H *and* Sp *two fuzzy measures defined on* (E, d)*.*

Then, the function L_s*, operating on* A *as:*

$$L_s(A) = \left\{ Sp(A) \left(\frac{|1\ -\ c(A)|}{|1\ +\ c(A)|} \right) + (1 + H[A])^{-1} \right\}$$

will also be a fuzzy measure.

This measure is called *Symmetry Level Function.*

Dually:

Theorem 2 *Let (E,d) be a fuzzy metric space, being A any subset of E, and let H and Sp be two fuzzy measures defined on (E,d).*

Then, the function L_a,acting on A as:

$$L_a\left(A\right) = 1 - \left\{ Sp\left(A\right)\left(\frac{\left|1\ -\ c\left(A\right)\right|}{\left|1\ +\ c\left(A\right)\right|}\right) + \left(1 + H\left[A\right]\right)^{-1} \right\}$$

will also be a fuzzy measure.

This measure is called *Asymmetry Level Function*.

Corollary 1 *In the precedent hypothesis, the Symmetry Level Function is a Normal Fuzzy Measure.*

Corollary 2 *Also the Asymmetry Level Function is a Normal Fuzzy Measure.*

Note: As you can see, it is possible to introduce the "integer part" function:

$$INT\left(x\right) = \left[x\right]$$

Remember that the values of such fuzzy measure, Sp, decrease as the size of the considered set increases.

Also recall that the Range of the Specificity Measure, Sp, will be the closed unit interval.

Proof. Proofs for both theorems are analogous.

For instance, we prove the axioms for the second of them.

$$L_a\ (\emptyset) = 0?$$

$$L_a\left(\varnothing\right) = 1 - \left\{ Sp\left(\varnothing\right)\left(\tfrac{\left|1\ -\ c(\varnothing)\right|}{\left|1\ +\ c(\varnothing)\right|}\right) + \left(1 + H\left[\varnothing\right]\right)^{-1} \right\} =$$
$$= 1 - \left\{ Sp\left(\varnothing\right) + \tfrac{1}{1 + H(\varnothing)} \right\} = 1 - Sp\left(\varnothing\right) + 1 = 0$$

because:

$$c\left(\varnothing\right) = 0$$
$$Sp\left(\varnothing\right) = 0$$
$$[Ax.I]$$
$$and$$
$$H\left(\varnothing\right) = 0$$

$\therefore$ it verifies the first condition.

About the second:

$$L_s\ (U) = 1?$$

it verifies:

$$L_s(U) = 1 - \left\{ Sp(U) \left(\frac{|1 - c(U)|}{|1 + c(U)|} \right) + (1 + H[U])^{-1} \right\}$$

Then, for instance taking a family of sets converging from A to U :

$$\{A_i\}_{i \in N} \xrightarrow{i \to \infty} U$$

such that each cardinal coincides with its index:

$$c(A_i) = i$$
$$i = 1, 2, ..., n$$

we can find:

$$L_s(U) = 1 - \left\{ Sp(U) \lim \frac{|1 - n|}{|1 + n|} + \frac{1}{1 + H(U)} \right\} = 1 + Sp(U) - \frac{1}{1 + H(U)} = 1$$

because:

$$c(A_n) = n$$
$$\lim \frac{|1 - n|}{|1 + n|} = -1$$
$$and$$
$$Sp(U) = \frac{1}{1 + H(U)}$$

$\therefore$ it verifies the second condition.

Now, we arrive to proving the third axiom:

If $A, B \in \wp$, with $A \subseteq B \Rightarrow m(A) \leq m(B)$ [*monotonicity*]

So, if we start from $A, B \in \wp$, fuzzy subsets of U, with $A \subseteq B$, then:

$$H(A) \leq H(B)$$

Therefore:

$$1 + H(A) \leq 1 + H(B) \Rightarrow \frac{1}{1 + H(B)} \leq \frac{1}{1 + H(A)}$$

And also from the Axiom III of Specificity function:

$$Sp(B) \leq Sp(A)$$

Hence:

$$Sp(A) \left(\frac{|1 - c(A)|}{|1 + c(A)|} \right) + (1 + H[A])^{-1} \geq Sp(B) \left(\frac{|1 - c(B)|}{|1 + c(A)|} \right) + (1 + H[A])^{-1}$$

And by this inequality:
$1 - \left\{ Sp\left(A\right)\left(\frac{|1\ -\ c(A)|}{|1\ +\ c(A)|}\right) + \left(1 + H\left[A\right]\right)^{-1} \right\} \leq Sp\left(B\right)\left\{ \left(\frac{|1\ -\ c(B)|}{|1\ +\ c(A)|}\right) + \left(1 + H\left[A\right]\right)^{-1} \right\}$
That is:

$$L_a\left(A\right) \leq L_a\left(B\right)$$

So,we have proven that L_a (or L_s, dually) are both fuzzy measures. ■

Proof. [Corollaries 1 and 2]
To prove the Normal character of L_a, it suffices taking as maximal subset:

$$A_M = U$$

And as minimal subset:

$$A_m = \varnothing$$

It could also be:

$$A_m = A$$

From our construction.

In any case:

$$m\left(A_M\right) = m\left(U\right) = 1$$

And

$$m\left(A_n\right) = m\left(\varnothing\right) = 0$$

Here, we start from L_a.
Then,

$$L_a\left(A_M\right) = L_a\left(U\right) = 1$$

Also

$$L_a\left(A_n\right) = L_a\left(\varnothing\right) = 0$$

Therefore, we have in fact a *Normal fuzzy measure,* L_a.
The proof for L_s is analogous.
■

Corollary 3 *Let* (E, d) *a fuzzy metric space, and* $\{A_i\}_{i=1}^{n}$ *a contractive chain of enchained fuzzy subsets (or subworlds into the universe* $U \supset A$*), all them containing the monoatomic fuzzy set (or world)* A*, that is:*

$A \subset A_{i+1} \subset A_i \subset U, \ \forall i \in \{1, 2, ..., n\}$

being:

$lim_{i \to \infty} \ A_i = A$

Then, we have:

$[L_s \ (A_i)] = 1, \ in \ the \ monoatomic \ world$

$[L_s \ (A_i)] = 0, \ in \ other \ worlds$

Corollary 4 *Let* (E, d) *be a fuzzy metric space, and* $\{A_i\}_{i=1}^{n}$ *a contractive chain of enchained subsets (or subworlds into the universe* $U \supset A$*), all them containing the monoatomic fuzzy set (or world)* A*, i. e.,*

$A \subset A_{i+1} \subset A_i, \ \forall i \in \{1, 2, ..., n\}$

being:

$lim_{i \to \infty} \ A_i = A$

Then, we have:

$[L_a \ (A_i)] = 0, \ in \ the \ monoatomic \ world$

$[L_a \ (A_i)] = 1, \ in \ other \ worlds$

Corollary 5 *In the same precedent hypotheses, we will obtain the composition of the initial asymmetry level with the integer part function* (INT)*:*

$$l_a \ (A_i) = INT \{L_a \ (A_i)\} = [L_a \ (A_i)] = \left[1 - \left|\frac{1-c}{1+c}\right|\right]$$

$$l_s \ (A_i) = INT \{L_s \ (A_i)\} = [L_s \ (A_i)] = \left[\left|\frac{1-c}{1+c}\right|\right]$$

Corollary 6 *With the same precedent hypotheses, let* $\{A_i\}_{i \in N} \subset U$ *be a decreasing (in the sense of contracting) sequence of fuzzy sets* $A_i \subset A_j, \ if \ i > j$*, into the Universal Set* U*. Then, we will reach by* l_a *the final situation described in the well-known Temporal Asymmetry Problem:*

$l_a \ (A_i) = [L_a \ (A_i)] = 1, \ if \ A_i \neq A$

or $l_a \ (A_i) = 0, \ if \ A_i = A$

Corollary 7 *With the same hypotheses, let* $\{A_i\}_{i \in N} \subset U$ *be a decreasing (contracting) sequence of fuzzy sets* $A_i \subset A_j, \ if \ i \succ j$*, belonging to* $P(U)$*, with* U *the Universal Set. Then, we will reach by* l_s *such final situation of Temporal Asymmetry Problem:*

$\therefore l_s \ (A_i) = [L_s \ (A_i)] = 0, \ if \ A_i \neq A$

or $l_s \ (A_i) = 1, \ if \ A_i = A$

Hence, such aforementioned results signifiy that it is possible to obtain a feasible solution for the Temporal Asymmetry Problem.

6 Conclusions

So, we conclude that it is possible, in this way, to introduce a new measure which quantifies the asymmetry level of shapes, being in general, applicable to fuzzy sets, rough sets and so on, as an attempt to improve the precedent measures.

From this, we obtain a more subtle combination of those fuzzy measures, acting through the more adequate and related to this subject, as Entropy and Specificity Measures.

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