Analysis of Asymmetry Measures

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Abstract

Some difficulties with the Theory of Lewis and its applications to Causal Analysis have appeared. Certain of them are concerned with Temporal Asymmetry and questions related with the mathematical treatment of Causality. Also it is a very classical problem in Physical theories, with its own characteristics.

Here, we analyze this problem and attemp to find a geometrical construct which also permits reaching an efficient measure of the Level of Asymmetry of Shapes and Graphs. So, for BNs (in particular) and Fuzzy Sets (in the more general case).

Keywords: Mathematical Modeling, Optimization, Fuzzy Measure Theory, Geometric Measure, Graph Theory, Bayesian Networks, Artificial Vision, Pattern Recognition, A. I.

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1. Introduction

The Counterfactual Theory starts with the work of the Scottish philosopher *David Hume (1711-1776).* Such initial theory is taken up again by *John Stuart Mill*, in 1843. Later, *David Kellogg Lewis (1941-2001)* developed successive, improved versions of their *Counterfactual Theory. Lewis* was an American mathematical logician and analytical philosopher. He worked on a great number of fields, lately developing successive improved versions of his *Counterfactual Theory.* In 1999, the last of such versions was issued: it was during the *Whitehead Lectures,* at Harvard University. In 1999 were exposed the last of such versions: it was into the *Whitehead Lectures,* at Harvard University.

In more recent times, it is essential the work of Judea Pearl (1936-).

We will previously analyze some of such ideas.

David Hume said in 1748: "We may define a cause to be an object followed by another, and where all the objects, similar to the first, are followed by an object similar to the second. Or, in other words, where, if the first object had not been, the second never had existed". The first paragraph reflects the *Regularity Criteria*. And the second is the known *Counterfactuality Criteria: "A has caused B",* which in counterfactual notation is denoted: $A \square\!\rightarrow B$, and is equivalent to: *"B would not have occurred, if it were not for A"*.

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Rather than a cumbersome enumeration of definitions, which may be found in good surveys in circulation, as [13], [5] or [17], we prefer showing certain illustrative quotations. Nevertheless, we introduce some definitions and commentaries when needed.

Pearl [13] writes: "Causality connotes lawlike necessity, whereas probability connotes exceptionality, doubt and lack of regularity ".

Lewis [10] asks: "Why not take counterfactuals as face value: as statements about possible alternatives to the actual situation?"

Implicitly, it is supposed that counterfactual expressions are less ambiguous, to our minds, that the causal ones.

Roughly speaking: to decide or not on counterfactuals requires generating and examining possible alternatives to the current situation, and also verifying whether certain propositions are valid on them.

And Pearl also asks: "How can this be done? What mental representation allows humans to process counterfactuals so swiftly and reliably, and what logic governs this process, so as to maintain uniform standards of coherence".

According to Lewis' explanation [11], for the evaluation of counterfactuals we need the notion of Similarity, which permits ordering possible worlds.

An *open question:* which election of Similarity Measure could be so that the counterfactual reasoning was compatible with the usual conceptions of cause and effect?

Furthermore, the Logic of Lewis establishes two rules and six axioms, being equivalent to the three axioms of Structural Counterfactuals (Composition, Effectiveness and Reversibility) on Recursive Systems. Such axioms hold on every Causal Model.

The supposition on Lewis, according to which: *an asymmetry of causal dependence characterizes our world* is basic in the Lewisian framework.

But there has been criticism against the explanation given by Lewis, in some authors, as [7], [14] or [6].

Our objective is to contribute to giving an answer to these objections, through the introduction of a new fuzzy measure, which reflects the asymmetry level of the system, whether it be Fuzzy Set, Graph, Bayesian Network or Markov Model.

The supposition on Lewis [11], according which: an asymmetry of causal dependence characterizes our world is basic into the Lewisian framework. But criticism appeared against the explanation given by Lewis, in some authors, as Horwich [7], Price [14] and Hausman [6].

One of the main arguments of the critics is based on supposing that this explanation of Lewis suffers from a certain psychological implausibility. This can be found in Horwich [7].

Lewis admits that this asymmetry is possibly a contingent characteristic of the actual world, not present in other worlds.

So, in a world populated by only one atom such asymmetry on the overdetermination does not hold. For this reason, there exists a possible discontinuity problem in the boundary. Because if we consider a contractive sequence of subworlds, each of them asymmetric, converging to the monoatomic world, denoted W, where asymmetry does not hold, we would have a weakness in the theory.

2. Causality and Symmetry

Remember now the properties of a Causality relation, or Causation.

Let A, B and C be three different events in a world, W.

Then, we have:

Transitivity: If A is cause of B, and B is cause of C, then A is cause of C.

Asymmetry or Anti-Symmetry: If A is cause of B, then B can not be the cause of A.

Irreflexivity or Anti-Reflexivity: A is not possible (never) to be the own cause.

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3. Symmetry vs Asymmetry

Symmetry is defined as *invariance under a specified group of transformations. Asymmetry* would be, then, the *absolute lack of symmetry.* And so, *Antisymmetry* relates to the apparition of *relative symmetry* in the properties of the observed object. Therefore, it corresponds to different degrees of symmetry in such properties, for the considered object.

In our world, there are many time asymmetries. Let us make mention of some:

Effects never seem to precede their causes.

A later state of higher entropy never follows another of lower entropy, according to the Second Principle of Thermodynamics. Therefore, entropy increases with time, in closed systems. It is maximal, for instance, in black holes. Note that the world acts spontaneously on the systems, maximizing entropy, or disorder, so minimizing potentials. Whereas the First Law makes no distinction of past, present and future, the Second Law introduces the basis for detecting this difference.

Humans have memories of their past, but never of their future.

Some of such asymmetries can be based on others. Among the thinkers with this opinion was David K. Lewis [11]: "the asymmetry of counterfactual dependence serves to explain more familiar asymmetries". This refers to temporal asymmetry of causation: a cause "*always*" precedes its effect.

Or the *asymmetry of openness:* "the obscure contrast we draw between the open future and the fixed past".

Lewis characterizes the asymmetry of counterfactual dependence in this manner: "the way the future depends counterfactually on the way the present is. If the present were different, the future would be different".

This supposition, according to which an asymmetry of causal dependence characterizes our world, is basic in the framework of Lewis' Theory.

Frequently, the causal relation is taken to be intrinsically asymmetric, because in the world of our experience it is so. However, the fundamental physical laws are symmetric. Any other temporal asymmetries are accounted in terms of the *Principle of the Common Cause* (PCC), due to Hans Reichenbach, which says: "If an improbable coincidence has occurred, there must exist a common cause"[16].

Through such Principle, it is possible to explain the arrows (of entropy, experience and so on) by the Causal Theory. And at the same time, the PCC results as Corollary of the Probabilistic Theory of Causation.

The *Entropic Theory* works in two phases: first, reducing any other arrow (causation, radiation, experience...) to the entropic arrow; and second, explaining entropic asymmetry in terms of boundary conditions on the universe.

Leyton [9] investigated the psychological relationships between shape and time, arguing that shape is used, by mind, to recover the past, and it forms a basis of the memory. And then, *symmetry* is the means by which shape is converted into memory.

Remind that the *Symmetry* is an intrinsic property which causes it to remain invariant under some classes of transformations. So, Rotation, Reflection, Inversion or more abstract mathematical operations.

For instance, it can be represented in the form of coefficients of equations.

In Physics we have the essential *Noether´s Symmetry Theorem*, given by great german mathematician Emmy Noether, in 1918: *each symmetry of a system leads to a physically conserved quantity.*

So, *Symmetry under Translation* corresponds to *Conservation of Momentum.*

Symmetry under Rotation corresponds to *Conservation of Angular Momentum.*

Symmetry in time, to *Conservation of Energy.*

And so on.

Indeed, Noether propose two theorems with many posterior consequences.

For the first, *there exist equivalence between an invariance property and a conservation law.*

For the second, *there exist relationships among an invariance and the existence of certain integral of the equations of motion.*

In recent times, some extensions of Noether theorem have been provided, in the mathematical literature, very useful for the solution of more general problems in Modeling and Optimization.

4. Measures of (A)Symmetry

From this point, after analyzing the "state of the art", we move on to some aspects of our initial contribution.

Usually, Symmetry and Asymmetry are considered two sides of the same coin: an object would be either totally symmetric, or totally asymmetric, in relation to a pattern object. Intermediate situations of partial symmetry or partial asymmetry are not taken into account. But this dichotomic classification, because of its simplicity, is lacking in necessary and realistic gradation. For this reason, it is convenient to introduce "shade regions", modulating the degree of symmetry (a fuzzy concept). So, we will describe the symmetry as a continuous feature, much more complex than the previous discrete definition, but more useful in many fields, as Computer Vision, in Laue photographs of X-ray beam spectra, where it allows analyzing the quality of crystalline structures according to their symmetry. Therefore, its interest is not only theoretical, but also applied, being possible to construct new plausible computational tools which permits the automatic transition from theoretical concepts on Symmetry/Asymmetry to applications in the real world. And with this, the apparition of a new collection of nearest shapes: because given an object O, we will define SD, the Symmetry Distance of the shape to its reference pattern.

This way, we are measuring the lack of symmetry in shape as a continuous feature, as opposed to a discrete one: gradual "similarity"of a shape to its symmetrical one, instead of either total coincidence or absolute difference.

This *distance from Symmetry in shape* is defined as the minimum mean squared distance required for moving points from the original shape, in order to obtain a symmetrical shape. So, *SD is the minimum effort required to turn a given shape into a symmetric shape.*

Every pair of such shapes (V and W, for instance) are represented by their respective sequence of points:

Then, the aforementioned metric, m, is defined as:

$$m : \Psi \times \Psi \to R_+ \cup \{0\}$$

$$m\,(V,\ W) = m\left(\{V_j\}_{j=0}^{j=n-1}\ ,\ \ \{W_j\}_{j=0}^{j=n-1}\right) =$$

$$= \frac{\|V_0 - W_0\|^2 + \|V_1 - W_1\|^2 + \ldots + \|V_{n-1} - W_{n-1}\|^2}{n} =$$

$$= \frac{\sum \|V_j - W_j\|^2}{n}$$

Also, we define the *Symmetric Transform of V*, denoted $ST(V)$, as the closest symmetric shape to V, relative to such metric.

By this tool, it is possible to introduce the *SD of a shape, V*, as the distance measured between such V and its Symmetry Transform, $ST(V)$.

We are going to show the *Algorithm* necessary to evaluate such *Symmetry Transform (ST):*

To start with, n original points:

$$\{V_j\}_{j=0}^{n-1}$$

which conform the shape of O_i.

First step:

Fold $\{V_j\}_{j=0}^{n-1}$ into $\left\{V_j^*\right\}_{j=0}^{n-1}$.

For instance, in the C_n case, rotating each point counter clockwise about the centroid by $2\pi\frac{j}{n}$ radians.

Second step:

Average out this new set of points:

$$V_0^{\diamondsuit} = \frac{1}{n}\sum_{j=0}^{n-1} V_j^*$$

Third step:

Unfold such average point, obtaining:

$$\left\{V_j^{\diamondsuit}\right\}_{j=0}^{n-1}$$

In the aforementioned example, of $C_n - symmetry,$ it consists in maintaining $V_0^{\diamondsuit}$, and then rotate the points $2\pi\frac{j}{n}$ radians. In this way, we can reach:

$ST\left(\{V_j\}_{j=0}^{n-1}\right) = \left\{V_j^{\diamondsuit}\right\}_{j=0}^{n-1}$

Corresponding one-to-one with the points of the precedent shape, but in a "more symmetrical "position now.

Therefore, the *SD of a shape* V is evaluated by passing first through its Symmetry Transform, and then computing their respective distance:

$$SD\ (V) = m\ (V,\ ST\,(V))$$

This measure is invariant under translation and rotation.

If the shape V is totally symmetric, then it coincides with its symmetric transform, and so SD is null.

Given a general shape, O, it is necessary the transformation which departing from its boundary, ∂O, goes to a finite sequence of points. This permits applying the precedent algorithm.

Such selection may proceed in different ways:

We can obtain a polyhedral (ever improved) approximation to O.

Suppose that ∂O is a closed planar curve of length L. Then, to introduce (for instance) five points:

$$\{V_i\}_{i=0}^{4}$$

it would be sufficient fixing an initial point, say V_0, and from here, applying a distance equal to $L/5$ over the curve, V_1, and so on, until V_4.

From then, turning out V_0 :

$$V_0\,(+\;L/5) \rightarrow V_1\,(+\;L/5) \rightarrow V_2\,(+\;L/5) \rightarrow$$
$$\rightarrow V_3\,(+\;L/5) \rightarrow V_4\,(+\;L/5) \rightarrow V_0$$

The difficulty can appear when the shapes are partially occluded, or perhaps the data set is noisy. In such case, a previous process of smoothing is required. For example, by the equiangular selection.

In a very common real-world situation: when the shape is partially occluded, we need to recompose the hidden region by our knowledge of its symmetrical features. It is possible to determine a centroid, which by succesive approximations, can give us their centre of symmetry.

The symmetry centre may be defined as the point which minimizes the total of symmetry distances:

$$min \sum SD$$

It is possible to locate it by iteratively applying a procedure of *hill-climbing*: the *gradient descent method.* For this, we depart from the centroid of the shape.

The position of each new point would be modelled by a Gausssian distribution, which by standardization can be considered a $N\,(0,1)$.

There also exists a method for evaluating such probable positions, given a set of measurements. Its theoretical basis is the *Maximum Likelihood Criterion.*

So, we can start off with n ordered points:

$$\{W_i\}_{i=0}^{n-1}$$

each one of them with locations described by a Gaussian:

$$W_i \sim N\,(V_i,\;\Lambda_i)$$
$$\forall i = 0, 1, 2, ..., n-1$$

being V_i their expected position and Λ_i the covariance matrix.

Finally, the probability distributions of SD values correspond to a *chi-square* $\left(\chi^2\right)$ distribution, with $(n-1)$ freedom degrees:

$$\chi^2_{n-1}$$

But, as known, this would be approximated by a Gaussian distribution.

5. Markovian Modeling

For each vertex or node, representing in the graph a random variable, we dispose of the probability distribution value associated with its position. So, each possible situation of the node, into the corresponding slice, must possess a numerical image of the random variable, that jointly with the symmetry distance value to the pattern object, O, provides of a pair, describing probabilistically its position and how far it is from its symmetrical final place. Because we do not know previously the exact position of each node in each slide, advancing trough the development structure. We only know the probability distribution of such position: with what non-deterministic value such node goes to fill a place.

It is possible to define a Markovian Decision Process from this model, as a sequential chain of steps to be carried through such randomized Markov process: each node only depends on the corresponding vertex that belongs to shapes either in the same or in the precedent slice (Markovian property).

Such shapes can be supposed n vertices-polyhedra in the first step, $\{V_j\}_{j=0}^{n-1}$, *n-1* vertices in the second shape, $\{V_j\}_{j=0}^{n-2}$, and so on, until reaching the triangular shape, $\{V_j\}_{j=0}^{2}$, the line, $\{V_j\}_{j=0}^{1}$ and finally, the monoatomic world: a point, $W = V_0$. All of such shapes would be included in its corresponding slice.

Furthermore, it is possible to suppose an associated asymmetry level, decreasing, by applying in each step the algorithm on its points to obtain the Symmetry Transform, before acting to delete the corresponding point:

$$\{V_j\}_{j=0}^{n-1} \to ST\left(\{V_j\}_{j=0}^{n-1}\right) \to \{V_j\}_{j=0}^{n-2} \to$$
$$\to ST\left(\{V_j\}_{j=0}^{n-2}\right) \to ... \to ST\left(\{V_j\}_{j=0}^{2}\right) \to$$
$$\to ST\left(\{V_j\}_{j=0}^{1}\right) \to ST\left(W = V_0\right) = V_0$$

The elimination order will be given by the natural decreasing order of the indices, according to the prefixed order of vertices in the original shape.

We can take as *Total Expectancy Reward (TER)*, for the minimization (instead of maximization) process the previously defined Symmetry Distance (SD) between the succesive shapes.

It is also possible to introduce a *new Reward function* as inversely proportional to such SD translated in 1:

$$TER = \frac{1}{1 + SD\ (O_i,\ O)}$$

In such a case, appliyng the procedure of maximization is logical, now without the final problem of discontinuity.

According to the observability of system states, we construct a *Fully Observable Markovian Decision Process (FOMDP),* described without hidden variables.

Associated with each step of this process, we have the "transition probabilities": in the instant t, the system is in the state S_i, after carrying out the action, or decision, a_i :

$$do(X = x_i)$$

When the system was in state S_{i-1}. Such probability of transition will be expressed as:

$$P_t\ \left(S_i\ /\ S_{i-1},\ a_i\right)$$

The underlying (and basic) idea is the replication of the shapes (therefore, the set of its nodes-vertices, representing random variables), on a sequence of temporal points. Because in our case, the random variables can be the succesive shapes, in the evolution, or their nodes.

Then, we reach a *Foliation of Bayesian Nets,* F , where each BN belongs to a temporal slice, and so the total construct will be a Dynamic Bayesian Net:

$$Foliation\ of\ BNs = S\,(T) = \cup_{t \in T}\ S\,(t)$$

Therefore, it contains its corresponding slices, $S\,(t)$, according the evolution of the system.

So, we can consider each shape immersed in their parallel hyperplane, into the global Foliation defined on BNs.

6. Entropy and Specificity Measures

Let U be the universe of discourse, with $\wp$ a $\sigma-$algebra on U.

Then, given a function

$$m : \wp \to [0,\ 1]$$

we describe m as a *Fuzzy Measure,* if it verifies:

I) $m\ (\emptyset) = 0$
II) $m\ (U)\ = 1$
III) If $A,\ B\ \in \wp$, with $A \subseteq B \Rightarrow$
$\Rightarrow m\ \ (A) \leq m\ \ (B)\ \ [monotonicity]$

When we take the *Entropy concept,* we attempt to measure the fuzziness, that is, the degree of being fuzzy for each element in $\wp$.

It can be designed as the function:

$$H : \wp \to [0,\ 1]$$

verifying:

I) If A is a crisp set $\Rightarrow H\,(A) = 0$
II) If $H\ \ (x) = 1/2,\ \forall x \in A \Rightarrow$
$\Rightarrow H\ \ (A)\ \ is\ maximal\ (total\ uncertainty)$
III) If A is *less fuzzified* than $B \Rightarrow$
$\Rightarrow H\ \ (A) \leq H\ \ (B)$
IV) $H\ \ (A) = H\ \ (U \setminus A)$

And the *Specificity Measure* will be introduced as a measure of the tranquillity when we take decisions. Such *Specificity Measure* is a function:

$$S\ p : [0,\ 1]^U \to [0,\ 1]$$

where:

I) $S\ p\ (\emptyset) = 0$
II) $S\ p\ (\varkappa) = 1 \Leftrightarrow \varkappa$ is a unitary set (singleton).
III) If ς *and* τ are normal fuzzy sets in U,
with: $\varsigma \subset \tau \Rightarrow S\ p\ (\varsigma) \geq S\ p\ (\tau)$.

Note: $[0,\ 1]^U$ denotes the class of fuzzy sets in U.

7. Asymmetry Level as a Normal Fuzzy Measure

We can define now the symmetry/asymmetry level so (using Specificity Measure):

$$L_a(A_n) = 1 - \{Sp(A_n)\}\left[\frac{|1 - c|}{|1 + c|}\right]$$
$$\therefore L_s(A_n) = \{Sp(A_n)\}\left[\frac{|1 - c|}{|1 + c|}\right]$$

From here, based on:

$$0 \leq Sp(A_n) \leq 1 \Rightarrow$$
$$\Rightarrow \lim\{L_s(A_n)\} = L_s(A) = L_s(\{a\})$$

About the intervention of the Entropy Measure, H, in our formula, we must consider that:

$$A_i \subseteq A_j \Rightarrow H(A_i) \leq H(A_j)$$

The Entropy degree increases when the cardinal (or number of elements, for finite sets) increases, and reciprocally:

$$1 + H(A_i) \leq 1 + H(A_j) \Rightarrow$$
$$\Rightarrow \frac{1}{1 + H(A_j)} \leq \frac{1}{1 + H(A_i)},$$
$$\textit{being } i \neq j$$

In our construction, we have: $j \geq i$

Also, we can obtain a more complete expression of the *Symmetry Level Function,* through the intervention of the Entropy Measure, H depending on the increment or decrement of the cardinal of the set:

$$L_s^* = \left\{Sp\left(\frac{|1 - c|}{|1 + c|}\right) + (1 + H)^{-1}\right\} \Leftrightarrow$$
$$\Leftrightarrow L_a^* = 1 - \left\{Sp\left(\frac{|1 - c|}{|1 + c|}\right) + (1 + H)^{-1}\right\}$$

Therefore, when H increase, L_s^* decrease (so, L_a^* increase).
And when Sp increase, L_s^* also increase (so, L_a^* decrease, respectively).

But actually, in these cases, we are back to cardinal two:

$$R_{INT} \in \{0,\ 1\}$$

Reappearing in such case the discontinuous situation for the described function.

It is also possible to formulate it (as we say) as function of the *Entropy, H*, because the value of such measure increases with the number of elements in the set.
Therefore:

$$\begin{gathered}[L_s^*(A_i)] = 1,\ \textit{in the monoatomic world} \\ \textit{and}\ [L_s^*(A_i)] = 0,\ \textit{in other worlds} \\ \therefore [L_a^*(A_i)] = 0,\ \textit{in the monoatomic world} \\ \textit{and}\ [L_a^*(A_i)] = 1,\ \textit{in other worlds}\end{gathered}$$

Obtaining so, finally, the composition of the initial asymmetry level with the integer part function (INT):

$$\begin{gathered}l_a^*(A_i) = INT\{L_a^*(A_i)\} = \left[1 - \left|\frac{1-c}{1+c}\right|\right] \\ l_s^*(A_i) = INT\{L_s^*(A_i)\} = \left[\left|\frac{1-c}{1+c}\right|\right] \\ l_a^*(A_i) = [L_a^*(A_i)] = 1,\ if\ A_i \neq A, \\ or \\ l_a^*(A_i) = 0,\ if\ A_i = A \\ \therefore l_s^*(A_i) = [L_s^*(A_i)] = 0,\ if\ A_i \neq A, \\ or \\ l_s^*(A_i) = 1,\ if\ A_i = A\end{gathered}$$

Finally, it is possible to omit the asterisk in the L' symbols.
So,we will obtain L_a or L_s, according to our purpose.

To prove the Normal character of L_a, it suffices taking as maximal subset:

$$A_M = U$$

And as minimal subset:

$$A_m = \varnothing$$

In any case:

$$m(A_M) = m(U) = 1$$

And

$$m(A_n) = m(\varnothing) = 0$$

Therefore, in fact we have a *Normal fuzzy measure*, L_a.

8. Related work and future research

So far, the fundamental mathematical direction when working on Symmetry and its properties was the geometrical, from problems derived from different fields. Let us cite some: the analysis of chrystalline stuctures, by the Chystallographic Planar or Spacial Groups; also, it was an immediate application of the classical Group Theory; and many more: physical problems, as in Quantum Mechanics, or in Penrose tiles, Fractals, Chaos Theory and so on.

Closer to the Computer Science, it is connected with Artificial Vision, Pattern Recognition, the analysis of symmetrical structures in Computational Linguistics or similar aspects of AI, where the presence or absence of certain symmetrical features, and their degrees, is essential.

Basically, the precedent work related to these aspects is on Symmetry Groups, the Hermann Weyl papers and his famous book, *Symmetry* [18]. About their application to pattern recognitition, artificial vision and so on, the papers and presentations of Liu on *Computational Symmetry* [12]. And also the revisited question of symmetrical patterns.

The future research needs to focus on questions derived from the versatility of the real world and the relatively coarse and rigid old geometry (group theory included), which only permits a first approximation to new and more difficult problems of AI, like in Computer Vision.

Conclusions

Our initial objective is finally reached, because we obtain a new fuzzy measure, which permits evaluating the degree of asymmetry, or dually, the degree of symmetry, of any fuzzy set. So, it is useful in general: it may be applied to graphs in general, or to Bayesian Networks in particular.

And many other applications in very promising and connected fields, as Modeling and Optimization.

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