Applications of rearrangements to nonlinear optimization problems

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Abstract

In this paper we consider two nonlinear optimization problems. We show that these problems can be reformulated in such a way that the admissible set becomes a subset of a prescribed rearrangement class. The well known theory of rearrangemnts is then applied to prove existence of optimal solutions. Numerical results determining optimal solutions are presented.

Key Words: Optimization, Rearrangements, Cost function, Length, Weak closure.

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1 Introduction

The theory of rearrangements as a tool for solving a variety of problems is growing in popularity. This theory has been established by Geoffrey R. Burton [Burton 1987, 1989]. Subsequently, many applications in solid and fluid mechanics [Burton et al., 1999; Emamizadeh et al., 2004], meteorology [Douglas, 2002] followed. In this paper, we demonstrate the use of rearrangements to prove the existence of optimal solutions to nonlinear optimization problems. For definiteness, we consider the following practical situation:

> *Suppose that a factory intends to run an electric machine for a fixed amount of time every day during the twenty four hour period. This task bears two types of costs; one for electricity and the other for labor. Given the electricity and labor cost distributions over the twenty four hour period, we would like to determine an optimal set of time intervals so that the total cost is maximal or minimal.*

To model the above problem we proceed as follows. Set $\Omega = [0, 24]$, and let g_1 and g_2 (the electricity and labor cost distributions, respectively) be two non-negative, bounded, and measurable functions defined on Ω. Denoting the set of all measurable subsets of Ω by 2^Ω, the *non-additive* total cost function $\mathcal{C} : 2^\Omega \to \mathbb{R}$ is defined as follows:

$$\mathcal{C}(F) = \left(\int_F g_1\, dx\right)^2 + \left(\int_F g_2\, dx\right)^2. \tag{1.1}$$

We are interested in the following optimization problems:

$$(M) \qquad \sup_{|F|=A} \mathcal{C}(F), \tag{1.2}$$

and

$$(m) \qquad \inf_{|F|=A} \mathcal{C}(F), \tag{1.3}$$

where $|F|$ denotes the one dimensional Lebesgue measure of F. We assume $0 < A < 24$, in order to avoid triviality.

2 Reformulation of (M) and (m); main results

Our first task is to reformulate (M) and (m) in terms of a rearrangement class. For this purpose we first observe that $\mathcal{C}(\cdot)$ can be written as follows:

$$\mathcal{C}(F) = \left(\int_\Omega \chi_F g_1\, dx\right)^2 + \left(\int_\Omega \chi_F g_2\, dx\right)^2,$$

where χ_F stands for the characteristic function of F. Next, since every set $E \subseteq \Omega$ can be identified with the characteristic function χ_E, there is a natural bijection between the two sets $\{F \subseteq \Omega : |F| = A\}$ and $\{\chi_F : \int_\Omega \chi_F dx = A\}$. Note that problems (M) and (m) are equivalent to:

$$(\tilde{M}) \qquad \sup_{\chi_F \in \{\chi_E : \int_\Omega \chi_E = A\}} \tilde{\mathcal{C}}(\chi_F),$$

and

$$(\tilde{m}) \qquad \inf_{\chi_F \in \{\chi_E : \int_\Omega \chi_E = A\}} \tilde{\mathcal{C}}(\chi_F),$$

respectively. Here we have

$$\tilde{\mathcal{C}}(f) = \left(\int_\Omega f g_1\, dx\right)^2 + \left(\int_\Omega f g_2\, dx\right)^2.$$

Clearly if $\chi_{\hat{F}}$ is a solution to $(\tilde{M})$, then $\hat{F}$ is a solution to (M). Similarly, if $\chi_{\hat{G}}$ is a solution to $(\tilde{m})$, then $\hat{G}$ is a solution of (m). Therefore the natural questions are whether $(\tilde{M})$ and $(\tilde{m})$ are solvable or not. Indeed, we show that the answers to both questions are affirmative as stated in the following theorems which are the main results of this paper.

Theorem 1 *The maximization problem $(\tilde{M})$ is solvable; that is, there exists $\chi_{\hat{F}}$, $|\hat{F}| = A$, such that*

$$\tilde{\mathcal{C}}(\chi_{\hat{F}}) = \sup_{\chi_F \in \{\chi_E : \int_\Omega \chi_E = A\}} \tilde{\mathcal{C}}(\chi_F).$$

Theorem 2 *Suppose* $g_1 = \sum_{k=1}^{N} \gamma_k^{(1)} \chi_{[a_k,b_k]}$, *and* $g_2 = \sum_{s=1}^{M} \gamma_s^{(2)} \chi_{[c_s,d_s]}$. *Then* $(\tilde{m})$ *is solvable; that is, there exists* $\chi_{\hat{G}}$, $|\hat{G}| = A$, *such that*

$$\tilde{\mathcal{C}}(\chi_{\hat{G}}) = \inf_{\chi_F \in \{\chi_E : \int_\Omega \chi_E = A\}} \tilde{\mathcal{C}}(\chi_F).$$

Remark 1. Note that the assumptions on g_i, $i = 1, 2$, in Theorem 2 are realistic. Indeed it is natural to expect the electricity and labor costs to be constants over various time intervals (sets).

3 Rearrangements

In this section we review some well known results from the theory of rearrangements of functions, as much as it is relevant to our purpose.

Recall that two functions $f, g : \Omega \to \mathbb{R}$ are said to be rearrangements of each other provided their respective distribution functions are equal; that is,

$$\lambda_f(\alpha) = \lambda_g(\alpha), \quad \forall \alpha \geq 0.$$

Here $\lambda_f(\alpha) = |\{x \in \Omega : f(x) \geq \alpha\}|$, and $\lambda_g(\alpha)$ is similarly defined. The reader is referred to [Hardy et al., 1988] for an extensive treatment of the distribution functions. When two functions f and g are rearrangements of each other we write $f \sim g$. Clearly if $f \sim g$, then $g \sim f$ as well. For a fixed non-negative function $f_0 : \Omega \to \mathbb{R}$, the rearrangement class generated by f_0 denoted $\mathcal{R}(f_0)$, is defined as follows:

$$\mathcal{R}(f_0) = \{f : \Omega \to \mathbb{R} : f \sim f_0\}.$$

If, in addition, we assume $f_0 \in L^\infty(\Omega)$, then the following result is well known, see for example [Burton 1987, 1989].

Lemma 1 *i) The set* $\mathcal{R}(f_0) \subseteq L^\infty(\Omega)$, *and* $\|f\|_\infty = \|f_0\|_\infty$, *for every* $f \in \mathcal{R}(f_0)$.
ii) The w^**-closure of* $\mathcal{R}(f_0)$, *in* $L^\infty(\Omega)$, *denoted* $\overline{\mathcal{R}}(f_0)$, *is* w^**-compact and convex.*
iii) Denoting the extreme values of $\overline{\mathcal{R}}(f_0)$ *by* $ext(\overline{\mathcal{R}}(f_0))$, $ext(\overline{\mathcal{R}}(f_0)) = \mathcal{R}(f_0)$.

Let us henceforth fix $F_0 \subseteq \Omega$ such that $|F_0| = A$. Then from [Turkington et al., 1989], we obtain:

Lemma 2 *With the notation introduced above we have:*

$$\overline{\mathcal{R}}(\chi_{F_0}) = \left\{ f : \Omega \to [0, 1] : \int_\Omega f dx = A \right\}.$$

4 Proofs of Theorems 1 and 2

Proof of Theorem 1. According to the discussion in the previous section we need to prove solvability of the following maximization problem:

$$\sup_{f \in \mathcal{R}(\chi_{F_0})} \tilde{\mathcal{C}}(f).$$

We first relax the problem by replacing $\mathcal{R}(\chi_{F_0})$ by $\overline{\mathcal{R}}(\chi_{F_0})$. Observe that $\tilde{\mathcal{C}}(\cdot)$ is w^*-continuous. To see this we consider $f_n \rightharpoonup f$, in $L^\infty(\Omega)$. Here "$\rightharpoonup$" indicates convergence with respect to w^*-topology in $L^\infty(\Omega)$. Since $g_i \in L^\infty(\Omega)$, it follows that $\int_\Omega f_n g_i dx \to \int_\Omega f g_i dx$, as n tends to infinity. Hence $\tilde{\mathcal{C}}(f_n) \to \tilde{\mathcal{C}}(f)$, as claimed. $\tilde{\mathcal{C}}(\cdot)$ being w^*-continuous, and $\overline{\mathcal{R}}(\chi_{F_0})$ compact, by Lemma 1, it follows that the relaxed problem:

$$\sup_{f \in \overline{\mathcal{R}}(\chi_{F_0})} \tilde{\mathcal{C}}(f)$$

is solvable. Let us assume $\overline{f}$ is a solution. To prove the solvability of the original problem it suffices to verify existence of $\hat{f} \in \mathcal{R}(\chi_{F_0})$ such that $\tilde{\mathcal{C}}(\hat{f}) = \tilde{\mathcal{C}}(\overline{f})$. To derive a contradiction we assume the contrary; that is,

$\tilde{\mathcal{C}}(f) < \tilde{\mathcal{C}}(\overline{f})$, for every $f \in \mathcal{R}(\chi_{F_0})$. From Lemma 1, we deduce that there exist $t \in (0,1)$, $f_i \in \mathcal{R}(\chi_{F_0})$, $i = 1, 2$, such that $\overline{f} = tf_1 + (1-t)f_2$. Observe that $\tilde{\mathcal{C}}(\cdot)$ is convex on $L^\infty(\Omega)$, so

$$\begin{aligned}\tilde{\mathcal{C}}(\overline{f}) = \tilde{\mathcal{C}}(tf_1 + (1-t)f_2) &\leq t\tilde{\mathcal{C}}(f_1) + (1-t)\tilde{\mathcal{C}}(f_2) \\ &< t\tilde{\mathcal{C}}(\overline{f}) + (1-t)\tilde{\mathcal{C}}(\overline{f}) \\ &= \tilde{\mathcal{C}}(\overline{f}),\end{aligned}$$

which is a contradiction, as desired. □

To prove Theorem 2, we need some preparations. We begin with defining a partition for Ω. To this end let us collect the right end points of the intervals $[a_i, b_i], [c_i, d_i]$, and form the set $S = \{b_1, \cdots, b_N, d_1, \cdots, d_M\}$. Next we arrange the elements of S in an increasing order:

$$e_1 < e_2 < \cdots < e_l,$$

where e_i equals some b_j or d_k; note also that $l \leq N + M$. Now define the *zones*, $Z_1, Z_2, \cdots, Z_l$ as follows:

$$Z_i = [e_{i-1}, e_i], \;\; i = 1, 2, \cdots, l,$$

where we set $e_0 = 0$. Note that $e_l = 24$, and $\Omega = \cup_{i=1}^{l} Z_i$. Let us state some properties of $C(\cdot)$ relative to the partition $\{Z_1, Z_2, \cdots, Z_l\}$:

Lemma 3 *The following statements hold for the set function $C(\cdot)$:*
a) For two sets E, F located in the same zone and satisfying $|E| = |F|$, we have $C(E) = C(F)$.
b) $C(\cdot)$ is additive on every zone; that is, if E and F are two disjoint subsets of Z_i, for some i, then $C(E \cup F) = C(E) + C(F)$.
c) For $E \subseteq Z_i$, $F \subseteq Z_j$, where $i \neq j$, we have

$$C(E \cup F) = C(E) + C(F) + P_{i,j}(E, F),$$

where $P_{i,j}(E, F) = 2\left(\int_E g_1 dx \int_F g_1 dx + \int_E g_2 dx \int_F g_2 dx\right)$.

Proof. All assertions follow easily from the very definition of C. □

Remark 3. A consequence of Lemma 3(a) is that if E is located in some Z_i, then $C(E) = C([\alpha, \beta])$, where $[\alpha, \beta] \subseteq Z_i$, and $\beta - \alpha = |E|$.

The following proposition shows that in case $l = 2$, the assertion in Theorem 2 is verifiable.

Proposition 1. *Suppose $l = 2$, then the minimization problem*

$$\inf_{f \in \mathcal{R}(\chi_{F_0})} \tilde{\mathcal{C}}(f) \tag{4.4}$$

is solvable. Proof. Since $l = 2$, there exist two zones Z_1 and Z_2. According to Remark 3, the minimization (4.4) reduces to finding $a \in [0, 24 - A]$ such that the interval $[a, a + A]$ minimizes $C(\cdot)$ relative to $\{F : |F| = A\}$. So, if we define $\xi : [0, 24 - A] \to \mathbb{R}$ by

$$\xi(a) = C([a, a + A]),$$

then (4.4) is equivalent to

$$\inf_{0 \leq a \leq 24 - A} \xi(a).$$

Since g_1 and g_2 are simple functions, it turns out that ξ is continuous, thus the above problem has a minimum, so $(\tilde{m})$ is solvable, which implies (4.4) is solvable. □

To state our next result we first define a partial ordering amongst $Z_1, \cdots, Z_l$.

Definition. We write $Z_i \preceq Z_j$, provided: For every $E \subset Z_i, F \subseteq Z_j$, satisfying $|E| = |F|$, we have $C(E) \leq C(F)$.

Proposition 2. *If $l = 2$, $Z_1 \preceq Z_2$, and $A \leq$ length (Z_1), then (4.4) is solvable. In fact any set $\hat{F} \subseteq Z_1$ with $|\hat{F}| = A$ is a solution* *Proof.* Fix $\hat{F} \subseteq Z_1$ satisfying $|\hat{F}| = A$. To verify $\hat{F}$ is a solution to (4.4) it suffices to

show $C(\hat{F}) \leq C(F)$. To do this we consider $F \subseteq \Omega$, so $F = F_1 \cup F_2$, where $F_1 \subseteq Z_1$ and $F_2 \subseteq Z_2$. Therefore by Lemma 3(c), we have

$$\begin{aligned} C(F) = C(F_1) + C(F_2) + P_{1,2}(F_1, F_2) &\geq C(F_1) + C(F_2) \\ &\geq C(F_1) + C(F_2'), \end{aligned}$$

where $F_2' \subset Z_1$, and $F_2' \cap F_1 = \emptyset$. Note that F_2' exists. Now applying Lemma 3(b) we infer $C(F_1) + C(F_2') = C(\hat{F})$. Thus, $C(F) \geq C(\hat{F})$, as desired. $\square$ *Proof of Theorem 2.* Using the ideas of Proposition 1, we should

look for $\hat{F}$ in the following form:

$$\hat{F} = [\delta_1^{(1)}, \delta_2^{(1)}] \cup \cdots\cdots \cup [\delta_1^{(l)}, \delta_2^{(l)}],$$

such that $[\delta_1^{(i)}, \delta_2^{(i)}] \subseteq Z_i$, $i = 1, \cdots, l$. Let us now introduce $\kappa : \mathbb{R}^{l\times l} \to \mathbb{R}$ by

$$\kappa(\delta_1^{(1)}, \delta_2^{(1)}, \cdots\cdots, \delta_1^{(l)}, \delta_2^{(l)}) = C(\hat{F});$$

it is clear that κ is a continuous piecewise quadratic polynomial in terns of $\delta_1^{(1)}, \delta_2^{(1)}, \cdots\cdots, \delta_1^{(l)}, \delta_2^{(l)}$. So $(\tilde{m})$ reduces to the following minimization problem

$$\inf_{\mathcal{A}} \kappa(\delta_1^{(1)}, \delta_2^{(1)}, \cdots\cdots, \delta_1^{(l)}, \delta_2^{(l)}), \tag{4.5}$$

where

$$\mathcal{A} = \left\{ (\delta_1^{(1)}, \delta_2^{(1)}, \cdots\cdots, \delta_1^{(l)}, \delta_2^{(l)}) : \ 0 \leq \delta_1^{(1)} \leq \delta_2^{(1)} \leq \cdots \leq \delta_1^{(l)} \leq \delta_2^{(l)}, \ \sum_{i=1}^{l} (\delta_2^{(i)} - \delta_1^{(i)}) = A \right\}.$$

Since $\mathcal{A}$ is a compact subset of $\mathbb{R}^{l\times l}$, it follows that $\inf_{\mathcal{A}} \kappa$ is solvable, hence $(\tilde{m})$ is. $\square$

5 When g_1 and g_2 are monotonic

In this section we discuss both $(\tilde{M})$ and $(\tilde{m})$ in case g_i are monotonic (increasing or decreasing).

Let us recall the following result due to Hardy and Littlewood, see for example [Hardy et al., 1988].

Lemma 4 *Suppose h_1 and h_2 are two non-negative integrable functions over Ω. Then the following inequalities hold:*

$$\int_\Omega (h_1)_\Delta h_2^\Delta dx = \int_\Omega h_1^\Delta (h_2)_\Delta dx \leq \int_\Omega h_1 h_2 dx \leq \int_\Omega h_1^\Delta h_2^\Delta dx = \int_\Omega (h_1)_\Delta (h_2)_\Delta dx.$$

Recall that h_Δ, h^Δ, respectively, denote the increasing, decreasing rearrangements of h.

Theorem 3 *(a) If g_1 and g_2 are both increasing on Ω, then $\chi_{\hat{F}}$ will be a solution of $(\tilde{M})$, and $\chi_{\overline{F}}$ a solution of $(\tilde{m})$. Here $\hat{F} = [24 - A, A]$ and $\overline{F} = [0, A]$. (b) If g_1 and g_2 are both decreasing on Ω, then $\chi_{\overline{F}}$ will be a*

solution of $(\tilde{M})$, and $\chi_{\hat{F}}$ a solution of $(\tilde{m})$. Here $\overline{F}$ and $\hat{F}$ are defined as in part (a).

Proof. We only prove part (a) since part (b) is similarly verified. Consider $F \subseteq \Omega$ satisfying $|F| = A$. Then by Lemma 4, we obtain

$$\begin{aligned} \tilde{\mathcal{C}}(\chi_F) &\leq \left(\int_\Omega (\chi_F)_\Delta (g_1)_\Delta dx \right)^2 + \left(\int_\Omega (\chi_F)_\Delta (g_2)_\Delta dx \right)^2 \\ &= \left(\int_\Omega \chi_{\hat{F}} g_1 dx \right)^2 + \left(\int_\Omega \chi_{\hat{F}} g_2 dx \right)^2 \\ &= \tilde{\mathcal{C}}(\chi_{\hat{F}}). \end{aligned}$$

So $\chi_{\hat{F}}$ is a solution of $(\tilde{M})$, as desired.

On the other hand,

$$\begin{aligned}
\tilde{\mathcal{C}}(\chi_F) &\geq \left(\int_\Omega (\chi_F)^\Delta (g_1)_\Delta dx\right)^2 + \left(\int_\Omega (\chi_F)^\Delta (g_2)_\Delta dx\right)^2 \\
&= \left(\int_\Omega \chi_{\overline{F}} g_1 dx\right)^2 + \left(\int_\Omega \chi_{\overline{F}} g_2 dx\right)^2 \\
&= \tilde{\mathcal{C}}(\chi_{\overline{F}}).
\end{aligned}$$

Thus, $\chi_{\overline{F}}$ is a solution of $(\tilde{m})$. □

6 Numerical Results

In this section we present numerical results that verify the theory. We solve the minimization problem (1.3) using the subroutine "wnnlp" from the public domain software package "wnlib". The algorithm in wnnlp uses techniques from operations research and the conjugate gradient method to determine the solution.

We divide the interval $[0, 24]$ into 3 zones corresponding to working and non-working hours $Z_1 = [0, 8]$, $Z_2 = [8, 17]$ and $Z_3 = [17, 24]$. We determine intervals $[x_1, x_2] \in Z_1$, $[x_3, x_4] \in Z_2$ and $[x_5, x_6] \in Z_3$ that yield the minimum cost given by (1.1) where $F = \chi_{[x_1,x_2]} \bigcup \chi_{[x_3,x_4]} \bigcup \chi_{[x_5,x_6]}$. In this context, we have the following constraints:

$$0 \leq x_1 \leq x_2 \leq 8 \leq x_3 \leq x_4 \leq 17 \leq x_5 \leq x_6 \leq 24,$$

$$x_2 - x_1 + x_4 - x_3 + x_6 - x_5 = A.$$

Example 1. We choose the costs g_i, $i = 1, 2$ to be monotonic increasing functions and constant in each zone; namely,

$$g_1(x) = \begin{cases} 5, & x \in Z_1 \\ 10, & x \in Z_2 \\ 20, & x \in Z_3 \end{cases} \qquad g_2(x) = \begin{cases} 10, & x \in Z_1 \\ 20, & x \in Z_2 \\ 30, & x \in Z_3 \end{cases}$$

The resulting cost function to be minimized is given by

$$\mathcal{C}(F) = [5(x_2 - x_1) + 10(x_4 - x_3) + 20(x_6 - x_5)]^2 + [10(x_2 - x_1) + 20(x_4 - x_3) + 30(x_6 - x_5)]^2.$$

On using wnnlp with $A = 7, 14, 21$, the solutions obtained are tabulated in Table 6. The solutions in the table affirm the result in Theorem 3.

A	x_1	x_2	x_3	x_4	x_5	x_6	min cost
7	0.50	7.50	15.33	15.33	22.97	22.97	6,125
14	0.00	8.00	11.00	17.00	22.17	22.17	50,000
21	0.00	8.00	8.00	17.00	19.01	22.99	188,091

Example 2. In this example, we choose the costs g_i, $i = 1, 2$ to be monotonic decreasing functions and constant in each zone, that is,

$$g_1(x) = \begin{cases} 20, & x \in Z_1 \\ 10, & x \in Z_2 \\ 5, & x \in Z_3 \end{cases} \qquad g_2(x) = \begin{cases} 30, & x \in Z_1 \\ 20, & x \in Z_2 \\ 10, & x \in Z_3 \end{cases}$$

The solutions when $A = 7, 14, 21$, tabulated in Table 6, affirm the result in Theorem 3.

Example 3. We choose the cost of electricity g_1 to be monotonic increasing and the cost of labor g_2 to be low during working hours and high during non-working hours; viz.,

$$g_1(x) = \begin{cases} 5, & x \in Z_1 \\ 10, & x \in Z_2 \\ 20, & x \in Z_3 \end{cases} \qquad g_2(x) = \begin{cases} 50, & x \in Z_1 \\ 20, & x \in Z_2 \\ 50, & x \in Z_3 \end{cases}$$

A	x_1	x_2	x_3	x_4	x_5	x_6	min cost
7	6.83	6.83	15.28	15.28	17.00	24.00	6,125
14	5.67	5.67	10.00	17.00	17.00	24.00	55,125
21	2.01	6.99	8.00	17.00	17.00	24.00	210,304

The solutions when $A = 7, 14, 21$ are tabulated in Table 6. We do not have theory for this case yet. From a practical point of view, the solutions obtained seem reasonable.

A	x_1	x_2	x_3	x_4	x_5	x_6	min cost
7	6.83	6.83	9.06	16.06	22.06	22.06	24,500
14	3.00	8.00	8.00	17.00	21.00	21.00	198,084
21	0.00	8.00	8.00	17.00	18.50	22.50	651,888

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