AMO – Advanced Modeling and Optimization, Volume, 10, Number, 2, 2008

MULTI-LEVEL MULTI-OBJECTIVE INTEGER LINEAR PROGRAMMING PROBLEM

Anuradha Gaur
Department of Mathematics, Maharaja Agrasen College, University of Delhi,
Delhi-110007, India
anuradhagaurs@yahoo.co.in

S.R. Arora
Department of Mathematics, Hans Raj College, University of Delhi,
Delhi-110007, India
srarora@yahoo.com

Abstract: This paper concentrates on Multi-level Multi-objective Decision-Making (MMDM) problem with linear constraints. The objective functions at each level are to be maximized and are linear functions. A convergent algorithm based on Stackelberg strategy is employed to solve the (MMILP) problem which does not increase the complexities of the problem considered here. It solves the problem hierarchically for a given choice of the variables under the control of the upper level decision maker (DM) and each level having several objectives which are conflicting in nature is solved by the weighted method by assigning a positive weight vector to each objective function and transforms it into a parametric program. An illustrative numerical example is given to demonstrate the algorithm.

Keywords: Parametric programming, Multi-objective optimization, Multi-level programming problem, Integer Programming.

AMO – Advanced Modeling and Optimization. ISSN: 1841-4311.

1. INTRODUCTION

Multilevel multiobjective optimization problems have attracted considerable attention from the scientific and economic community in recent years. The multilevel multiobjective system has extensive existences in management fields. Usually, this kind of problem can be solved by multiple mathematical programming. Most of studies in this field are focused on bilevel programming [2,3,4,8]. However, many practical problems need to be modelled as multilevel multiobjective program evolving new appropriate and efficient methods. Multilevel Multiobjective Integer Linear Programming (MMILP) problems involve sequential or multistage decision making [5, 9]. An MMILP problem concerns with decentralized planning problems with multiple decision makers (DMs) in a multilevel or hierarchical organization where decision makers have interacted with each other. Multilevel Multiobjective Programming Problem is computationally more complex than the conventional Multi-Objective Programming Problem (MOPP) or a Multi-Level Programming Problem (MLPP).

One of the important characteristics of Multi-Level Programming Problems (MLPP) is that a planner at a certain level of hierarchy may have his/her objective function and decision space determined partially by other levels. Further, the control instruments of each planner can affect the policies at other levels to improve his/her own objective function. These instruments may include the allocation and use of resources at lower levels and the advantages obtained from other levels. MLP problems share the following common features.

1. The system has interacting decision making units within a hierarchical structure.
2. Each subordinate level performs its policies after knowing completely the decisions of superior levels.
3. Each unit maximizes net benefits independently of other units but may be influenced by actions and reactions of those units.
4. The external effect on a decision maker's problem can be reflected in both the objective function as well as on the set of feasible decisions.

The MMILP problem considered in this paper has K decision makers located at K different hierarchical levels, each independently controls a set of decision variables and each DM has q ($q \geq 2$) objective functions at each level. The hierarchical nature of the problem is reflected by the order imposed on the choice of the decision. One level makes his/her decision according to that of his/her higher level. It is assumed that the DM at first level, DMI, called the Leader, masters the information of the follower's objectives and constraints, while the followers make their decisions after the leader's strategy is announced. The DMs at lower level also have effect on upper objective function, while DMs at upper level may adjust their decision until their objective functions are satisfied. The decision makers at the same level have common constraints and make decision cooperatively.

Due to the complexity of the MMLP problems, there exists no efficient traditional techniques for obtaining the solutions of the problem with reasonable size. The decision deadlock arises in some situations due to rejecting the solution by the followers for not giving a decision power to it. In the techniques for solving the MMILP problems, the decisions of all the DMs are sequential alongwith essential cooperation with each other to make a balance of decision powers to the DMs. These methods have been introduced primarily to tackle situations when the DM has no prior information on the desired levels for the several objective functions and on the priorities and the ranking as in goal programming. They are also pertinent when no information is available on the weights indicating their relative importance.

The techniques used for solving multilevel multiple objective integer linear programming problems are diverse : cutting plane techniques, dynamic programming approaches, dual simplex procedures, branch and bound algorithms or iterative techniques that consist of solving a sequence of progressively more and more constrained linear / non-linear programs with single objective. The structure of MMILP problem being complex rarely admits of a globally optimal solution to the MMILP problems.

As a class of MLPP [2, 8], most of the developments focus on bi-level linear programming. Anandalingam [1] studied bi-level non-linear programming. Bi-level multi-objective with multiple interconnected decision makers was discussed in [4]. Several three – level programming problems along with their solution methods were studied and introduced viz. hybrid extreme – point search algorithm [3]. A bibliography of the related references on bi-level and multi-level programming in both linear and non-linear cases, which is updated biannually, can be found [7].

The basic concept of the MLPP technique is that the first level decision maker (DMI) sets his goal/decision, then asks each subordinate level of the organization for their optima which are calculated in isolation. The lower level DMs are then submitted and modified by DMI in consideration of the over all benefit for the organization. The process continues until a compromise solution is reached. In this paper we propose a new method for solving a multilevel multiobjective system. It has more extensive application in practice and is based on the weighted method approach for the solution of MOPPs. The proposed algorithm is inspired by the work of Crema and Sylva [6] for the multiobjective integer linear programs. As is generally the case, passing from MOILP to MMILP is not trivial. In this paper, we focus on the problem of optimizing a linear function over the efficient set of a MMILP problem. For each level of the MMILP, a direct approach could consist of finding all efficient solutions of the MMILP problem and then optimizing the corresponding parametric programming problem on that set. But this approach is not appropriate for practical purposes. We hereby propose an implicit technique that avoids search for all efficient solutions. Motivated by the concept of parametric programming, we would like to examine the possibility of unifying the level-wise (hierarchical) operation and stagewise operation for the MMILP problem. The advantage of the proposed method is that the DMs progressively learn about the problem, the nature and the conflict among the objectives and the solution process. This paper is organized as follows : Section 2 presents the MMILP problems mathematically. Section 3 features the definitions and theoretical development of the problem alongwith determining the efficient solutions to the leader's problem. Section 4 explains the solution technique and termination condition. Section 5 introduces the algorithmic representation of the solution technique and a flow diagram.

2. FORMULATION OF MMILP PROBLEMS

In the MMILP under consideration, in order to arrive at a solution which is acceptable to all the decision makers they would be required to cooperate with each other to make a balance of decision powers. For attaining this solution, they may compromise by giving a possible relaxation of their individual Pareto-optimal decision. In such a case, the K objective functions F_1, F_2,...,F_K at different levels are each transformed into the corresponding parametric programs hierarchically by means of assigning an imprecise weight vector to each of them

Let the hierarchical system be comprised of K levels of decision makers, where the higher level decision maker, called the leader, controls decision variables $\overline{X}_1 = \left\{x_1^1, x_1^2 \dots x_1^{n_1}\right\}$ and the lower level divisions control decision variables

$$\bar{X}_j = \left\{x_j^1, x_j^2 \ldots x_j^{n_j}\right\}, \quad 2 \le j \le K$$

The overall system is described by a set of constraints which provide a feasible set $S \subset E^{n_1+n_2+\ldots n_k}$ for $\bar{X}_j, 1 \le j \le K$, where E^n denotes the n-dimensional Euclidean space. Let

DM1 denote the DM at the first (Upper) level,

DM2 denote the DM at the second level,

$\vdots$

DMK denote the DM at the K^{th} level.

We can formulate a maximization K-level MMILP problem mathematically as follows :

$$\text{(MMILP)} \qquad \text{DM1} \ \max_{\bar{X}_1} F_1(\bar{X})$$

$$\text{DM2} \ \max_{\bar{X}_2} F_2(\bar{X})$$

$$\vdots$$

$$\text{DMK} \ \max_{\bar{X}_K} F_K(\bar{X})$$

subject to

$$\bar{X} \in S$$

where, $\bar{X} = \bar{X}_1 \cup \bar{X}_2 \cup \ldots.. \cup \bar{X}_K$; $n = n_1 + n_2 + \ldots + n_K$;

$S = \left\{X : AX = b, X \ge 0, X \in \mathbb{Z}^n\right\}$; $b \in \mathbb{Q}^m$; $\bar{X} \in \mathbb{Q}^n$; $A \in \mathbb{Q}^{m \times n}$; $\mathbb{Z}$ is the set of integers; $\mathbb{Q}$ is the set of rational numbers; $\bar{X}_j$ is the decision vector under the control of DM_j $(1 \leq j \leq K)$ and has n_j number of decision variables; F_j $(1 \leq j \leq K)$ is the objective function at the j-th level defined as

$F_i(X) = C^i X = (C^{i1}X, C^{i2}X, \ldots C^{iq}X)$ $(q \geq 2)$ each $C^{iq}X$ being a row vector for $1 \leq i \leq K, q \geq 2$;

$\bar{X}_1 = \left\{x_1^1, x_1^2, ..., x_1^{n_1}\right\}^T$ is (are) decision variable(s) under control of DM1 (leader),

$\bar{X}_2 = \left\{x_2^1, x_2^2 \ldots x_2^{n_2}\right\}^T$ is (are) decision variable(s) under the control of DM2,

$\vdots$

$\bar{X}_K = \left\{x_K^1, x_K^2 \ldots x_K^{n_K}\right\}^T$ is (are) decision variable(s) under the control of DMK

with q decision makers on each level $(q \geq 2)$, n decision variables and m constraints.

It is assumed that S is a non-empty and bounded set over the convex polyhedron. The DM at the rth level where r = 1, 2, …., K individually solves his/her maximization problem and the DMs at the same level carry same status for executing their decision powers in the decision making situation. In real practice, due to the conflicting objectives, there is not a maximum solution for each level, but an efficient solution.

3. NOTATIONS, DEFINITIONS AND THEORETICAL DEVELOPMENT

The problem (L_u) $(1 \leq u \leq K)$ is defined as

$$\max \left\{C^u X : AX = b, X \geq 0, X \in \mathbb{R}^n\right\}$$

and the problem (L'_u) is defined as

$$\max\left\{\lambda^T C^u X : AX = b, X \geq 0, X \in \square^n\right\}.$$

Definition 1 : A point $X^o \in S$ is said to be an efficient point of a real valued function F defined on S if and only if there does not exist another point $X' \in S$ such that

$$F(X^1) \geq F(X^o)$$

with strict inequality holding atleast once.

If there exists a point X^1 for which

$$F(X^1) \geq F(X^o)$$

with strict inequality holding atleast once, then the point X^0 is said to be dominated by X^1 and $F(X^0)$ is said to be a dominated vector.

Definition 2: A point $X^0 = \left\{x_0^1, x_0^2, \ldots, x_0^{n_1}, x_0^{n_1+1}, \ldots, x_0^{n_2}, \ldots, x_0^{n_K} = x_0^n\right\}$ is an efficient point to the problem (MMILP) if and only if

(a) X^o is an efficient point to the Leader's problem.

(b) $\{x_0^{n_1+1}, \ldots, x_0^n\}$ is an efficient solution to the problem (L_2) for a given $\left\{x_0^1, \ldots x_0^{n_1}\right\}$,

$\vdots \qquad \vdots$

(c) $\{x_0^{n_{(K-1)}+1}, \ldots x_0^n\}$ is an efficient solution to the problem (L_K) for a given $\{x_0^1, \ldots, x_0^{n_{K-1}}\}$

Steuer [5] proposed the following theorem which connects the Multiple Objective Programming and Parametric Programming:

Theorem: *If X^* is an optimal solution to the parametric programming problem*

$$\max \{\lambda^T CX : X \in S\}$$

for some $\lambda \in \mathbb{R}^q, \lambda > 0,$ *then X^* is an efficient solution to problem*

$$\max \{CX : X \in S\}$$

The converse of the theorem stated above does not hold for Multiobjective Integer Linear Programming Problems as it might be the case that some efficient solutions may not be optimal for any $\lambda > 0$. However, it is possible to find new efficient solutions if known efficient solutions are removed from the feasible set.

The efficient solution of each level is determined by the method described in the theorem mentioned above for a given value of the variables under the control of the upper level decision maker (DM).

We now present a procedure for finding efficient solutions to the Leader's problem (DM1) using parametric programming based on the technique of Sylva and Crema [6] defined as follows:

Choose a weight vector $\lambda > 0$ and solve the following Integer Linear Programming (ILP) problem:

$$(L_1'): \ \max \{\lambda^T C^1 X : Ax = b, X \geq 0, X \in \mathbb{Z}^n\}$$

If there exists no solution for (L_1'), then the problem (MMILP) is unfeasible.

If (L_1') admits of an optimal solution $\overline{X}^1 = (\overline{x}^1, \overline{x}^2, ..., \overline{x}^n)$, then it is an efficient solution to the problem (L_1) defined as

(L_1): $\max\{C^1X : AX = b, X \geq 0, X \in \Re^n\}$

In order to find the other efficient solutions to (L_1), a sequence of progressively more constrained problems (LP_ℓ) is solved.

In this manner, a new efficient solution X^ℓ is determined.

Deleting all solutions from the feasible set of $(LP_{\ell-1})$ such that $C^1X \leq C^1X^\ell$, a new problem (LP_ℓ) is defined by adding the following linear constraints to the problem $(LP_{\ell-1})$:

$$(C^1X)_r \geq ((C^1X^\ell)_r + 1)y_r^\ell - M_r(1 - y_r^\ell), \text{ for } r = 1, 2, \ldots, q$$

$$\sum_{r=1}^{K} y_r^\ell \geq 1$$

$$y_r^\ell \in \{0, 1\} \text{ for } r = 1, 2, \ldots, q$$

where $-M_r$ is a lower bound of the r-th objective function of the leader for any feasible value of the objective function.

Note that each time addition of these constraints is equivalent to truncating the region

$$N_t = \{X \in \Re^n : C^1X \leq C^1X^t\} \qquad (1 \leq t \leq \ell)$$

from the feasible set the problem (LP_ℓ) is equivalent to the problem (PN_ℓ) defined by

$$(PN_\ell): \qquad \max\left\{\lambda^T C^1 X : X \in S - \bigcup_{t=1}^{\ell} N_t\right\}$$

where S is the feasible set of the (MMILP) problem.

Also, any solution to the problem (PN_ℓ) is efficient to problem (LP_ℓ) defined as:

(LP_ℓ): max $\lambda^T C^1 X$

subject to

$Ax = b$

$(C^1X)_r \geq (C^1X^t)_r + 1)y_\ell^t - M_r(1-y_r^t)$, for $t = 1, 2, \ldots, \ell$; $r = 1, 2\ldots, q$

$\sum_{r=1}^{q} y_r^t = 1, \quad y_r^t \in \{0,1\}$ for $t = 1, \ldots, \ell$; $r = 1, \ldots, q$,

$X \geq 0, X \in \mathbb{R}^n$

The procedure will produce the whole set of non-dominated vectors if all the elements of the matrix C^1 are integers.

4. METHOD OF SOLUTION

4.1 Solution Technique

In this paper we have proposed new method for the solution of the problem considered above in which the solution is obtained hierarchically and levelwise. Stackelberg strategy has been employed as a solution concept. Firstly DM1 optimizes his/her objective function by using the parametric approach. Then for a given value of the variable(s) under the control of DM1, DM2 optimizes his/her objective function by parametric programming method. If DM2 also produces the same solution, we move to the next level, otherwise, we find the next efficient solution to the leader's problem by adding a set of linear constraints. The process is continued till the last level and each time the next efficient solution to DM1 is obtained by progressively adding a set of constraints. The solution so obtained is the solution to the (MMILP) problem. Although, addition of the constraints to the leader's problem increases the problem size, but the algorithm proposed in this paper usually solves a multilevel multiobjective integer linear programming problem easily in a finite number of iterations and does not increase the complexities of the original problems. Each decision maker optimizes his/her objective function independently without producing any harm to the choice of the decision variables at the lower succeeding levels.

4.2 Termination condition of the iterative process for MMILP Problems

When the efficient solution of each decision maker considered hierarchically for a given choice of the variables under the control of upper level decision maker is same, the termination condition is satisfied.

5. ALGORITHM AND FLOWCHART

The outline of the procedure is summarized in the following algorithm and flowchart:

5.1 Technical representation of the algorithm:

The algorithm to solve the (MMILP) considered in this paper is technically summarized as follows:

Step 0: **Initialization**

Consider the leader's problem i.e. problem of first level decision maker DM1

(L_1) : $\max C^1X$

subject to

$$AX = b$$

$$X \geq 0$$

$$X \in \square^n$$

Step 1: **Solving the problem (L_1):** Choose a weight vector $\lambda > 0$ and solve the integer linear programming problem (L_1').

(L_1') $\max \{\lambda^T C^1 X : AX = b,\ X \geq 0,\ X \in \square^n\}$

If the problem (L_1') is infeasible, then the problem (MMILP) is unfeasible.

Otherwise, let $\hat{X}^1$ be the optimal solution of the problem (L_1'). Then, according to theorem 1, the solution $\hat{X}^1$ so obtained is an efficient solution to the leader's problem (L_1).

Step 2: Set u = 2, i.e. Move to the next level.

Consider the problem (L_u) defined as

(L_u) : $\max C^uX$

subject to

$AX = b$

$X \geq 0$

$X \in \square^n$

for a given $(\overline{x}^1, \overline{x}^1, \ldots, \overline{x}^{n(u-1)})$

Find the efficient solution of (L_u) by solving the corresponding parametric problem (L'_u) defined as:

(L'_u) : max $\{\lambda^T C^u X : AX = b, X \geq 0, X \in \square^n\}$

for a given $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^{n(u-1)})$

Let X^u be an optimal solution of the problem (L'_u).

Step 3 : If $X^u = \hat{X}^1$, set u = u + 1 and repeat step 2 and continue the process until we find an efficient solution of a multi level programming problem.

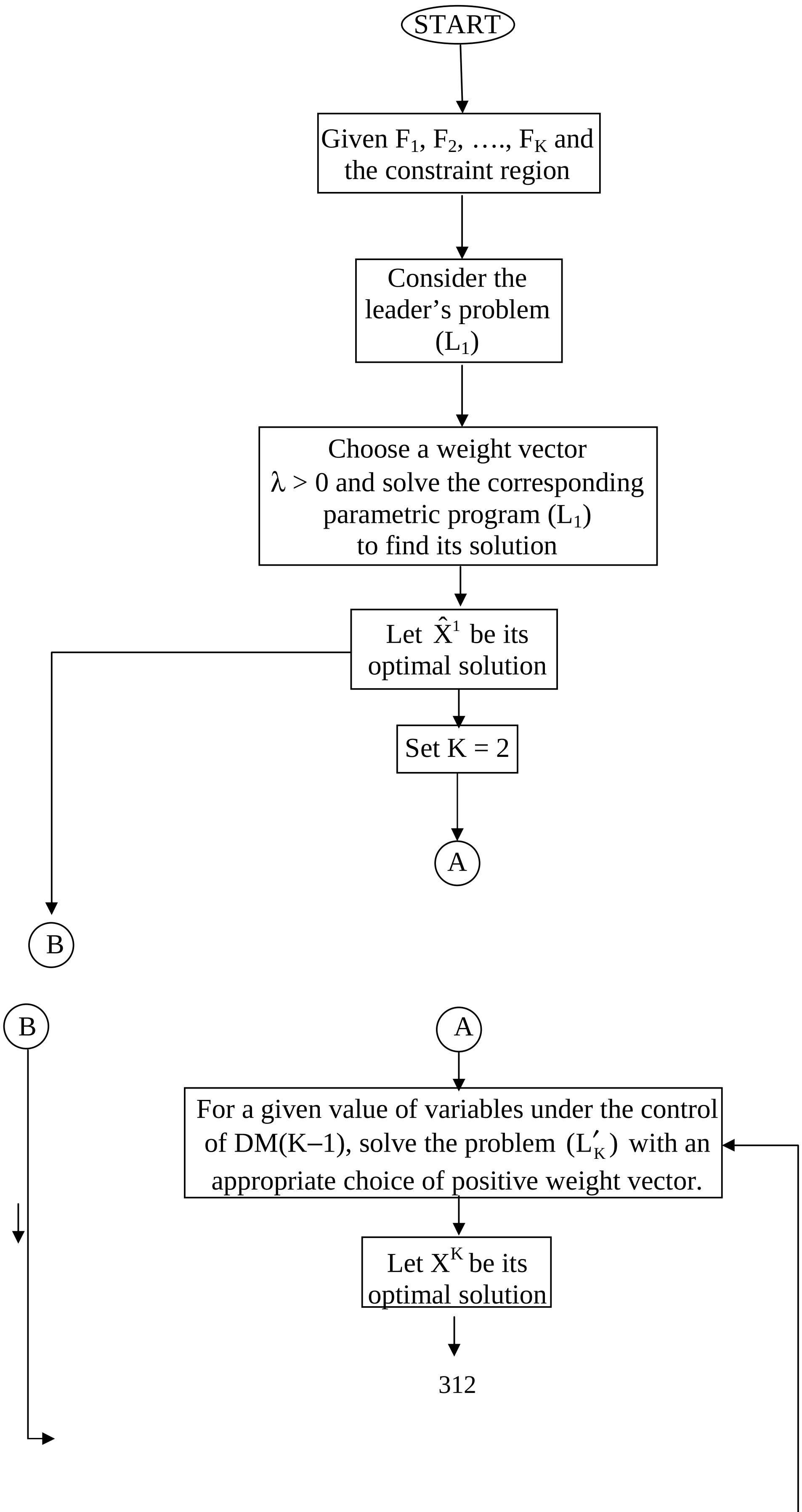
If $X^u \neq \hat{X}^1$ for any u, then we find the next efficient solution of (L'_1) and repeat the process for the second, third …., uth level until a solution is obtained which is the solution of multilevel programming problem.

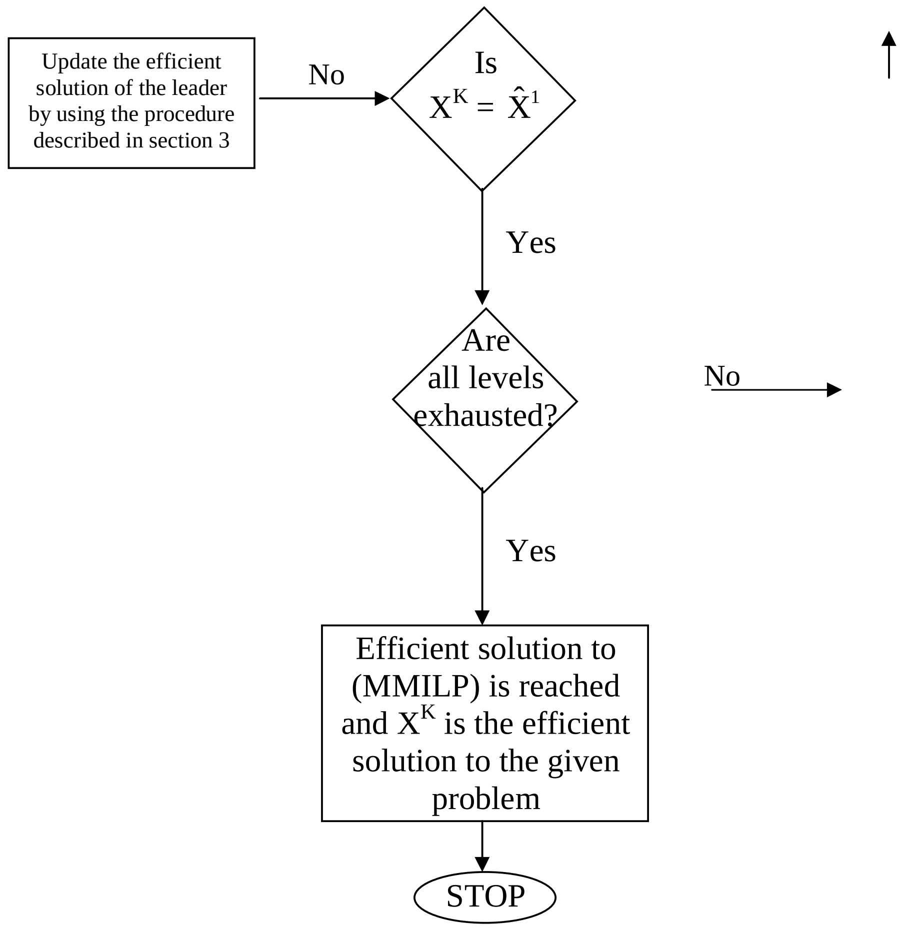
This stops the algorithm.

Step 4: $\hat{X}^u$ is the efficient solution of the given problem (MMILP) where $\hat{X}^u$ is the u-th efficient solution of the leader’s problem.

5.2 Flow Chart

The flow chart of the algorithm is as follows





6. NUMERICAL EXAMPLE

Consider the following (MMILP) problem:

$$\max_{x_1,x_2}(2x_1 - x_2 - x_3 + 4x_4 + x_5,\ x_1 + 3x_2 + 3x_3 - x_5)$$

$$\max_{x_3}(x_1 - 2x_2 - 2x_3 + x_4, -x_2 + 2x_3 - 3x_5)$$

$$\max(2x_1 - 2x_2 - x_3 - x_4 + 2x_5, - x_1 + x_2 + 5x_4 - 3x_5)$$

subject to

$$3x_1 - x_2 + 3x_3 \qquad + 4x_5 \le 6$$

$$-x_1 + 2x_2 - x_3 + 5x_4 + 4x_5 \le 9$$

$$-x_1 \qquad + 5x_3 + 4x_4 + 2x_5 \le 8$$

$$5x_1 \qquad - x_4 + 3x_5 \le 7$$

$$-x_1 \qquad -x_3 + 2x_4 \qquad \le 7$$

$x_i \in \{0, 1, 2\}$ for $i = 1, 2, 3, 4, 5$.

$x_1, x_2, x_3, x_4, x_5 \ge 0$

Step 0 : Consider the leader's problem

$$(L_1) : \max_{x_1, x_2}(2x_1 - x_2 - x_3 + 4x_4 + x_5,\ x_1 + 3x_2 + 3x_3 - x_5)$$

subject to

$$3x_1 - x_2 + 3x_3 \qquad + 4x_5 \le 6$$

$$-x_1 + 2x_2 - x_3 + 5x_4 + 4x_5 \le 9$$

$$-x_1 \qquad + 5x_3 + 4x_4 + 2x_5 \le 8$$

$$5x_1 \qquad - x_4 + 3x_5 \le 7$$

$$-x_1 \qquad -x_3 + 2x_4 \qquad \le 7$$

$x_i \in \{0, 1, 2\}$ for $i = 1, 2, 3, 4, 5$.

$x_i \ge 0$ for $i = 1, 2, 3, 4, 5$

Step 1 : **Solving the Problem (L_1)**

Choose λ = (4, 1) and solve the integer linear programming (ILP) problem (L_1') defined as

$$\max_{x_1, x_2} 4(2x_1 - x_2 - x_3 + 4x_4 + x_5) + (x_1 + 3x_2 + 3x_3 - x_5)$$

$= 9x_1 - x_2 - x_3 + 16x_4 + 3x_5$

subject to

$$\begin{aligned}
3x_1 - x_2 + 3x_3 \quad\quad + 4x_5 &\le 6 \\
-x_1 + 2x_2 - x_3 + 5x_4 + 4x_5 &\le 9 \\
-x_1 \quad + 5x_3 + 4x_4 + 2x_5 &\le 8 \\
5x_1 \quad\quad - x_4 + 3x_5 &\le 7 \\
-x_1 \quad -x_3 + 2x_4 \quad &\le 7
\end{aligned}$$

$x_i \in \{0, 1, 2\}$ for i = 1, 2, 3, 4, 5.

$x_i \ge 0$ for i = 1, 2, 3, 4, 5

The optimal value of the objective function is –3 at the point $\hat{X}^1 = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5)$ = (0, 2, 1, 0, 0) with objective vector (–3, 9) at $\hat{X}^1$.

Step 2 : Set u = 2

i.e. consider the problem (L_2) defined as

max C^2X

subject to

$$Ax = b$$

$$X \geq 0$$

$$X \in \square^n$$

for a given $(\hat{x}_1, \hat{x}_2)$

i.e. consider

$$(L_2): \quad \max_{x_3} C^2 X = (x_1 - 2x_2 - 2x_3 + x_4, -x_2 + 2x_3 - 3x_5)$$

$$= (-4 - 2x_3 + x_4, -2 + 2x_3 - 3x_5)$$

subject to

$$3x_3 + 4x_5 \leq 8$$

$$-x_3 + 5x_4 + 4x_5 \leq 5$$

$$5x_3 + 4x_4 + 2x_5 \leq 8$$

$$-x_4 + 3x_5 \leq 7$$

$$-x_3 + 2x_4 \leq 7$$

$x_i \in \{0, 1, 2\}$ for i = 1, 2, 3, 4, 5.

$x_3, x_4, x_5 \geq 0$

Choose $\lambda = (6, 5)$ and solve the integer linear programming (ILP) problem (L_2') defined as

$$(L_2'): \quad \max_{x_3} -34 - 2x_3 + 6x_4 - 15x_5$$

subject to

$$3x_3 + 4x_5 \leq 8$$

$$-x_3 + 5x_4 + 4x_5 \le 5$$

$$5x_3 + 4x_4 + 2x_5 \le 8$$

$$-x_4 + 3x_5 \le 7$$

$$-x_3 + 2x_4 \le 7$$

$x_3, x_4, x_5 \in \{0, 1, 2\}$

$x_3, x_4, x_5 \ge 0$

The optimal value of the objective function is -28 at

$X^2 = (0, 2, 0, 1, 0)$

Step 3 : Since $X^2 = (0, 2, 0, 1, 0) \ne (0, 2, 1, 0, 0)$, therefore we find the next efficient solution of (L_1) by adding a set of linear constraints to the problem (L_1) to obtain

$$\max_{x_1, x_2} 9x_1 - x_2 - x_3 + 16x_4 + 3x_5$$

subject to

$$3x_1 - x_2 + 3x_3 + 4x_5 \le 6$$

$$-x_1 + 2x_2 - x_3 + 5x_4 + 4x_5 \le 9$$

$$-x_1 + 5x_3 + 4x_4 + 2x_5 \le 8$$

$$5x_1 - x_4 + 3x_5 \le 7$$

$$-x_1 - x_3 + 2x_4 \le 7$$

$$2x_1 - x_2 - x_3 + 4x_4 + x_5 \ge -2y_1^1 - 3(1 - y_1^1) = y_1^1 - 3$$

$$x_1 + 3x_2 + 3x_3 - x_5 \ge 10y_2^1 - 1(1 - y_2^1) = 11y_2^1 - 1$$

$$y_1^1 + y_2^1 \geq 1$$

$$y_1^1,\ y_2^1 \in \{0,1\}$$

$$x_i \in \{0, 1, 2\} \quad \text{for } i = 1, \ldots, 5$$

Note that –3 is a lower bound on the first objective and –1 is a lower bound on the second objective of the leader.

The problem value of the objective function to this problem is 41 at $\hat{X}^2 = (1, 0, 0, 2, 0)$ with $y_1^1 = 1,\ y_2^1 = 0$ and the corresponding objective vector is (10, 1)

Step 4 : For a given value of $x_1 = 1$ and $x_2 = 0$, we solve the second level problem defined as

$$\max_{x_3} (1 - 2x_3 + x_4,\ 2x_3 - 3x_5)$$

subject to

$$\begin{aligned}
3x_3 \qquad\quad + 4x_5 &\leq 3 \\
-x_3 + 5x_4 + 4x_5 &\leq 10 \\
5x_3 + 4x_4 + 2x_5 &\leq 9 \\
-x_4 + 3x_5 &\leq 2 \\
-x_3 + 2x_4 \qquad\quad &\leq 8 \\
x_3, x_4, x_5 &\in \{0, 1, 2\} \\
x_3, x_4, x_5 &\geq 0
\end{aligned}$$

Choose $\lambda = (2, 1)$ and solve the following ILP problem

$$\max_{x_3} \; 2 - 2x_3 + 2x_4 - 3x_5$$

subject to

$$\begin{aligned}
3x_3 \quad\quad + 4x_5 &\leq 3 \\
-x_3 + 5x_4 + 4x_5 &\leq 10 \\
5x_3 + 4x_4 + 2x_5 &\leq 9 \\
-x_4 + 3x_5 &\leq 2 \\
-x_3 + 2x_4 \quad\quad &\leq 8 \\
x_3, x_4, x_5 &\in \{0, 1, 2\} \\
x_3, x_4, x_5 &\geq 0
\end{aligned}$$

The optimal value of the objective function is 6 at the point $(1, 0, 0, 2, 0) = \hat{X}^2$. Hence we move to the third level problem.

Step 5 : We now solve the problem of DM3 for a given value of $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ i.e. the problem (L_3) defined as

$$(L_3'): \quad \max_{x_4, x_5} \; (2 + x_4 + 2x_5 \,,\, -1 + 5x_4 - 3x_5)$$

subject to

$$
\begin{aligned}
x_5 &\le 3 \\
5x_4 + 4x_5 &\le 10 \\
4x_4 + 2x_5 &\le 10 \\
-x_4 + 3x_5 &\le 2 \\
2x_4 \qquad &\le 8 \\
x_4, x_5 &\in \{0, 1, 2\} \\
x_4, x_5 &\ge 0
\end{aligned}
$$

Choose $\lambda = (2, 1)$ and solve the integer linear programming problem defined as

$$(L_3'): \quad \max_{x_4, x_5} \; (-5 + 7x_4 + x_5)$$

subject to

$$
\begin{aligned}
x_5 &\le 3 \\
5x_4 + 4x_5 &\le 10 \\
4x_4 + 2x_5 &\le 10 \\
-x_4 + 3x_5 &\le 2 \\
2x_4 \qquad &\le 8 \\
x_4, x_5 &\in \{0, 1, 2\} \\
x_4, x_5 &\ge 0
\end{aligned}
$$

The optimal value of the objective function is 14 at $x_4 = 2$, $x_5 = 0$ which is same as $\hat{X}^2$. Hence, $\hat{X}^2 = (1, 0, 0, 2, 0)$ is the efficient solution of the given (MMILP) problem.

7. CONCLUSIONS

The main advantage of the proposed hierarchical and levelwise approach is that positive weight vectors are assigned to each level of the hierarchical system thereby transforming the problem of each decision maker into a parametric programming problem which can be solved efficiently. Moving from one efficient solution to the other efficient solution of the leader (DM1) is sequential and progressively more constrained, but does not increase the complexities of the problem. The algorithm employs Stackelberg strategy as a solution concept. Furthermore, for large scale problems, the method is efficient and flexible enough to produce a useful set of solutions. It is hoped that the proposed approach can contribute to future study in the field of practical hierarchical decision making problems.

8. REFERENCES

1. Anandalingam, G. A mathematical programming model of the decentralized multi-level systems, Journal of Operations Research Society **39(11)** (1988) 1021-1033.
2. Ayed O.B., Bilevel linear programming, Computers and Operations Research, **20(5)** (1993) 485-501.

3. Bialas W.F. and Karwan M.H., Two level linear programming, Management Science **30(18)** (1984) 1004-1020.
4. Shi X., Xia H., Model and interactive algorithm of bi-level multi-objective with multiple interconnected decision makers, Journal of Multicriteria Decision Analysis, **10** (2001) 27-34.
5. Steuer R.E., Multiple Criteria Optimization – Theory, Computation and Application, John Wiley and Sons (1986).
6. Sylva, J., Crema A., A method for finding the set of non-dominated vectors for multiple objective integer linear-programs, **158** (2004) 46-55.
7. Vicent L.N., Calamai, P.H., Bilevel and Multilevel programming: a bibliography review, Journal of Global Optimization, **5** (1994) 291-306.
8. Wen U.P., Lin S.F., Finding an efficient solution to linear bi-level programming problem: an efficient approach, Journal of Global Optimization, **8** (1996) 295-306.
9. White D.J., Optimality and Efficiency, John Wiley & Sons (1982).